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PRECONDITIONED HSS-LIKE ITERATIVE METHOD FOR SADDLE POINT PROBLEMS*

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Abstract

A new HSS-like iterative method is first proposed based on HSS-like splitting of non-Hermitian (1,1) block for solving saddle point problems. The convergence analysis for the new method is given. Meanwhile, we consider the solution of saddle point systems by preconditioned Krylov subspace method and discuss some spectral properties of the preconditioned saddle point matrices. Numerical experiments are given to validate the performances of the preconditioners.

Mathematics subject classification: 65F10, 65F50.
Key words: Saddle point problem, Non-Hermitian positive definite matrix, HSS-like splitting, Preconditioning.

1. Introduction

We consider the solution of the following saddle point linear system

$$\mathcal{A}\mathbf{x} = \begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = \mathbf{b}, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian positive definite matrix, that is, the matrix $H = (A+A^*)/2$, the Hermitian part of A, is positive definite, $B \in \mathbb{C}^{m\times n}$, with $m \leq n$, has full row rank. Such linear systems arise in a large number of scientific computing and engineering applications (see for instance [11-12, 18-21, 25-26, 31-32, 34]). As such systems are typically large and sparse, iterative methods become more attractive than direct methods for solving the saddle point problem (1.1). Solution by iterative methods can be found in the literature, such as Uzawa-type schemes [16, 18, 36], SOR-like and GSOR iterative methods [14, 16, 31, 37], matrix splitting methods [1-4, 6-14, 17, 23-24, 27-30, 33], iterative projection methods [35], restrictively preconditioned conjugate gradient (RPCG) methods [5, 15] and iterative null space methods [18], and so on.

In [6], Bai, Golub and Ng presented an Hermitian and skew-Hermitian splitting (HSS) method for solving non-Hermitian positive definite linear systems. The use of HSS as a stationary iteration for solving saddle point systems has been proposed in [2-3, 9, 13], where it

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was shown that the iteration converges for a large class of problems. Bai, Golub and Ng [8, 13] further generalized HSS to positive-definite and skew-Hermitian splitting (PSS), normal and skew-Hermitian splitting (NSS) and considered preconditioners based on these splittings. Pan, Ng and Bai [28 proposed two preconditioners for the saddle point problem with a non-Hermitian positive definite (1,1) block A, using the HSS and PSS of A, not based on using of the coefficient matrix $\mathcal{A}$ as a preconditioner for Krylov subspace methods. Recently, Jiang and Cao [23] presented a local Hermitian and skew-Hermitian iterative method and analyzed the convergence of the LHSS method. Zhang, Ren and Zhou [33] also presented an HSS-based constraint preconditioner, in which the (1,1) block of the preconditioner is constructed by the HSS method for solving the non-Hermitian positive definite linear systems.

In this paper, we propose a new HSS-like iterative method for the saddle point problem (1.1) based on the HSS of the (1,1) block A. We mainly focus on the case that A is a non-Hermitian positive definite matrix with the Hermitian part. We first establish a new HSS-like iterative method for the saddle point problem (1.1) and then give the convergence analysis of the new method in Section 2. In Section 3, we will show that the HSS-like iteration can provide an effective preconditioner for Krylov subspace methods applied to (1.1). Meanwhile, we present a modified HSS-like preconditioner and give spectral analysis of the preconditioned matrix. Numerical experiments are presented in Section 4. Meanwhile, we draw some conclusions.

2. The New HSS-like Iteration Method

From now on, we will adopt the general notation

$$\mathcal{A} = \begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} \tag{2.1}$$

to represent the non-Hermitian saddle point matrix of Eq. (1). We assume that A is non-Hermitian positive definite, and that B is of size $m\times n$ and has full row rank. Let $H = (A+A^*)/2$ and $S = (A - A^*)/2$ be its Hermitian and skew-Hermitian parts.

Let $\alpha > 0$ be a parameter, and consider the following splitting of A,

$$A = M_\alpha - N_\alpha = \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S) - \frac{1}{2\alpha}(\alpha I - H)(\alpha I - S).$$

Note that A is non-Hermitian positive definite. Then $M_\alpha = \frac{1}{2\alpha}(\alpha I+H)(\alpha I+S)$ is nonsingular. Thus we make the following special splitting:

$$\begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S) & 0 \\ -B & Q \end{bmatrix} - \begin{bmatrix} \frac{1}{2\alpha}(\alpha I - H)(\alpha I - S) & -B^* \\ 0 & Q \end{bmatrix},$$

where Q is a Hermitian positive definite matrix. We propose a new iterative method based on this special splitting.

Given an initial guess $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$, the new HSS-like iteration is given as follows:

$$\begin{bmatrix} M_\alpha & 0 \\ -B & Q \end{bmatrix} \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} N_\alpha & -B^* \\ 0 & Q \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix},$$

or equivalently, it can be written as

$$\begin{cases} x_{k+1} = x_k + M_\alpha^{-1}(f - Ax_k - B^* y_k), \\ y_{k+1} = y_k + Q^{-1}(Bx_{k+1} + g). \end{cases} \tag{2.2}$$