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ON THE OPTIMAL CONVERGENCE RATE OF A ROBIN-ROBIN DOMAIN DECOMPOSITION METHOD*

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Abstract

In this work, we solve a long-standing open problem: Is it true that the convergence rate of the Lions' Robin-Robin nonoverlapping domain decomposition (DD) method can be constant, independent of the mesh size h? We closed this old problem with a positive answer. Our theory is also verified by numerical tests.

Mathematics subject classification: 65N30, 65M60.
Key words: Finite element, Robin-Robin domain decomposition, Convergence rate.

1. Introduction

Domain decomposition (DD) methods are important tools for solving partial differential equations, especially by parallel computers. In this paper, we shall study a class of nonoverlapping DD method, which is based on using Robin-Robin boundary conditions as transmission conditions on the subdomain interface. The idea of employing Robin-Robin coupling conditions in DD methods was first proposed by P.L. Lions in [24]. In the past twenty years, there have been many works on the analysis and applications of this DD method: Despres [8], Douglas and Huang [12,13], Deng [6,7], Du [14], Gander et al. [20,21], Guo and Hou [22], Discacciati [9], Flauraud and Nataf [16], Gander [17,19], Qin and Xu [26-28], Discacciati et al. [10], Lui [25], and Chen et al. [2,3]. We should say that the list is far from being complete.

By comparison with other DD methods, Lions' DD method has several advantages. The iterative procedure is simple and much more highly parallel than others. Because it employs Robin conditions, the method is specially suitable for solving Helmholtz and time-harmonic Maxwell equations. There exists a lot of works in this direction, cf. [1,8,11,21] for details.

Lions' Robin-Robin DD method was proposed in 1990 [24], see Definition 1.1 below (without Step 5). The convergence (without any rate) is shown in [24,29]. Later, the convergence was improved to a geometric convergence [12,13,22], i.e, a rate of $1-O(h)$. It was first pointed out by Gander, Halpern and Nataf in [20] that the optimal choice of relaxation parameter is $\gamma = O(h^{-1/2})$ and the convergence rate $1-O(\sqrt{h})$ could be achieved. Recently, Xu and Qin [30]

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give a rigorous analysis on this result and shows that the rate is asymptotically sharp. However, without enough knowledge on the method, the two parameters γ_1 and γ_2 in Lions' DD method are set equal, see Definition 1.1 below, by researchers in above references. Thus, the rate of $1-O(\sqrt{h})$ is generally believed optimal for the Lions' DD method.

This paper answers this long-standing open problem: Is it possible to achieve a rate of $1-C$ for some some constant $C>0$ independent of the mesh size h? We give a positive answer. Yes, the constant rate of convergence is achieved by well-choosing three parameters in the Robin-Robin DD method, γ_1, γ_2 and θ, in Definition 1.1. Roughly speaking, the optimal choices are

$$\gamma_1 = O(1),\ \gamma_2 = O(h^{-1}),\quad \text{and}\quad \theta = \frac{2t-1}{2t+1},$$

where $t \approx 1$ is the ratio of spectral radii of two Dirichlet-Neumann operators on two subdomains. It is shown in this paper, by three types of analysis, that the error reduction rate of the DD method is optimal, $1-C$.

Next, we introduce the Robin-Robin DD method through a simple model problem. We solve the following model problem in 2D, which is decomposed into two subproblems (cf. Fig. 1.1):

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega_1, \\ u = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_1, \\ u - w = \dfrac{\partial u}{\partial \mathbf{n}} - \dfrac{\partial w}{\partial \mathbf{n}} = 0 & \text{on} \quad \Gamma, \\ -\Delta w = f & \text{in} \quad \Omega_2, \\ u = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_2, \end{cases} \tag{1.1}$$

where Γ is an interface separating Ω_1 and Ω_2, and $\mathbf{n}$ is an outward normal vector of Ω_1 at Γ. The DD method can be applied to general elliptic PDEs, general domains and multiple subdomains, cf. [6, 29].

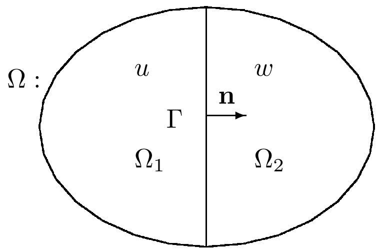

Fig. 1.1. A domain is decomposed into two subdomains.

The Dirichlet and Neumann interface conditions on Γ in (1.1) are combined into two Robin interface conditions:

$$\gamma_1 u + \frac{\partial u}{\partial \mathbf{n}} = \gamma_1 w + \frac{\partial w}{\partial \mathbf{n}} = g_1 \qquad \text{on} \quad \Gamma, \tag{1.2}$$

$$\gamma_2 u - \frac{\partial u}{\partial \mathbf{n}} = \gamma_2 w - \frac{\partial w}{\partial \mathbf{n}} = g_2 \qquad \text{on} \quad \Gamma. \tag{1.3}$$

Here we allow γ_1, γ_2 to be any positive constants. For example, when γ_1 is arbitrarily close to zero and γ_2 is close to infinity (but the linear systems would become near singular), the