Journal of Computational Mathematics
Vol.32, No.4, 2014, 392–411.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1403-m4373

# SPECTRAL AND SPECTRAL ELEMENT METHODS FOR HIGH ORDER PROBLEMS WITH MIXED BOUNDARY CONDITIONS*

Benyu Guo
*Department of Mathematics, Shanghai Normal University, Shanghai 200234, China*
*Scientific Computing Key Laboratory of Shanghai Universities, Division of Computational Science of E-institute of Shanghai Universities, Shanghai 200234, China*
*Email: byguo@shnu.edu.cn*
Tao Sun
*Department of Applied Mathematics, Shanghai Finance University, Shanghai 201209, China*
*Email: taosun80@yeah.net*
Chao Zhang
*Department of Mathematics, Jiangsu Normal University, Xuzhou 221116, China*
*Email: chaozhang@jsnu.edu.cn*

**Abstract**

In this paper, we investigate numerical methods for high order differential equations. We propose new spectral and spectral element methods for high order problems with mixed inhomogeneous boundary conditions, and prove their spectral accuracy by using the recent results on the Jacobi quasi-orthogonal approximation. Numerical results demonstrate the high accuracy of suggested algorithm, which also works well even for oscillating solutions.

*Mathematics subject classification:* 65L60, 65M70.
*Key words:* Spectral and spectral element methods, High order problems with mixed inhomogeneous boundary conditions.

## 1. Introduction

The Legendre and Chebyshev spectral methods have been used successfully for numerical solutions of differential equations, see, e.g., [3, 6, 7, 11, 12, 24] and the references therein. Some authors also developed the Jacobi spectral method of singular differential equations, see, e.g., [13, 14, 20]. Guo et al. [17, 18] proposed the generalized Jacobi spectral method enlarging the applications. We considered second order problems mostly. However, it is also important to deal with high order problems arising in science and engineering, see, e.g., [2,4,5,8,15,22,25-27] and the references therein. Recently, Guo et al. [19] developed the generalized Jacobi quasi-orthogonal approximation, which generalize the Legendre quasi-orthogonal approximation given in [16, 21, 23]. In particular, it leads to the probability of producing new spectral and spectral element methods for high order problems with various mixed boundary conditions.

In this work, we investigate new numerical methods for high order differential equations, by using the recent results on the Jacobi quasi-orthogonal approximation. The next section is for preliminaries. In Section 3, we propose spectral element method for high and even order problems with mixed inhomogeneous Dirichlet-Neumann boundary conditions, and prove its spectral accuracy. We also provide the spectral element method with essential imposition of mixed boundary conditions. In Section 4, we consider spectral method for high and odd order

* Received March 26, 2013 / Revised version received February 10, 2014 / Accepted March 25, 2014 / Published online July 3, 2014 /

problem and prove its spectral accuracy in space. In Section 5, we present some numerical results demonstrating the effectiveness of suggested algorithm, which also works well even for oscillating solutions. The final section is for concluding remarks. Although we only considered two model problems in this paper, the idea and techniques developed in this work open a new goal for designing and analyzing spectral and spectral element methods of many other high order problems with various boundary conditions.

## 2. Preliminaries

Let $\Lambda = \{x \mid |x| < 1\}$ and $\chi(x)$ be a certain weight function. For any integer $r \geq 0$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ as usual, with the inner product $(u,v)_{r,\chi}$, the semi-norm $|v|_{r,\chi}$ and the norm $\|v\|_{r,\chi}$. In particular, we denote the inner product and the norm of $L^2_\chi(\Lambda)$ by $(u,v)_\chi$ and $\|v\|_\chi$, respectively. The space $H^r_{0,\chi}(\Lambda)$ stands for the closure in $H^r_\chi(\Lambda)$ of the set $\mathcal{D}(\Lambda)$ consisting of all infinitely differentiable functions with compact support in $\Lambda$. We omit the subscript $\chi$ in notations, whenever $\chi(x) \equiv 1$.

Let $\chi^{(\sigma,\lambda)}(x) = (1-x)^\sigma(1+x)^\lambda$, $\sigma, \lambda > -1$, and $J^{(\sigma,\lambda)}_l(x)$ be the Jacobi polynomials of degree $l$. For any integers $m, n \geq 1$, we set

$$Y^{(m,n)}_l(x) = (1-x)^m(1+x)^n J^{(m,n)}_{l-m-n}(x), \qquad l \geq m+n.$$

The set of all polynomials $Y^{(m,n)}_l(x)$ is a complete $L^2_{\chi^{(-m,-n)}}(\Lambda)$-orthogonal system, namely (see (2.9) of [18]),

$$\int_\Lambda Y^{(m,n)}_l(x)Y^{(m,n)}_{l'}(x)\chi^{(-m,-n)}(x)dx = \gamma^{(m,n)}_l \delta_{l,l'}, \tag{2.1}$$

where $\delta_{l,l'}$ is the Kronecker symbol, and

$$\gamma^{(m,n)}_l = \frac{2^{m+n+1}\Gamma(l-m+1)\Gamma(l-n+1)}{(2l-m-n+1)\Gamma(l+1)\Gamma(l-m-n+1)}, \qquad l \geq m+n.$$

For any $v \in L^2_{\chi^{(-m,-n)}}(\Lambda)$, we have

$$v(x) = \sum_{l=m+n}^{\infty} \hat{v}^{(m,n)}_l Y^{(m,n)}_l(x), \tag{2.2}$$

where

$$\hat{v}^{(m,n)}_l = \frac{1}{\gamma^{(m,n)}_l}\int_\Lambda v(x)Y^{(m,n)}_l(x)\chi^{(-m,-n)}(x)dx.$$

For any positive integer $N$, $\mathcal{P}_N(\Lambda)$ stands for the set of all algebraic polynomials of degree at most $N$, and

$$\mathcal{Q}^{(m,n)}_N(\Lambda) = \text{span}\{Y^{(m,n)}_l(x),\ m+n \leq l \leq N\}.$$

The projection $P_{N,m,n} : L^2_{\chi^{(-m,-n)}}(\Lambda) \to \mathcal{Q}^{(m,n)}_N(\Lambda)$ is defined by

$$(P_{N,m,n}v - v, \phi)_{\chi^{(-m,-n)}} = 0, \qquad \forall \phi \in \mathcal{Q}^{(m,n)}_N(\Lambda). \tag{2.3}$$

For numerical solutions of high order differential equations, we need other orthogonal projections. For this purpose, we introduce the space

$$H^r_{m,n,A}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } ||v||_{H^r_{m,n,A}} < \infty\},$$

equipped with the following semi-norm and norm,

$$|v|_{H^r_{m,n,A}} = ||\partial_x^r v||_{\chi^{(-m+r,-n+r)}}, \qquad ||v||_{H^r_{m,n,A}} = \left(\sum_{k=0}^{r} |v|^2_{H^k_{m,n,A}}\right)^{\frac{1}{2}}.$$

If $v \in H^r_{m,n,A}(\Lambda), m,n \geq 1$ and $r \geq \max(m,n)$, then there exist finite traces $\partial_x^k v(-1)$ for $0 \leq k \leq n-1$, and $\partial_x^k v(1)$ for $0 \leq k \leq m-1$. Accordingly, we define the space

$$H^r_{0,m,n,A}(\Lambda) = \Big\{ v \in H^r_{m,n,A}(\Lambda) \mid \partial_x^k v(-1) = 0 \text{ for } 0 \leq k \leq n-1, \text{ and } \partial_x^k v(1) = 0 \text{ for } 0 \leq k \leq m-1 \} \Big\}.$$

For integer $\max(m,n) \leq \mu \leq m+n$, the projection $P^{\mu,0}_{N,m,n} : H^{\mu}_{0,m,n,A}(\Lambda) \to \mathcal{Q}^{(m,n)}_N(\Lambda)$ is defined by

$$\Big(\partial_x^{\mu}(v - P^{\mu,0}_{N,m,n} v), \partial_x^{\mu}\phi\Big)_{\chi^{(-m+\mu,-n+\mu)}} = 0, \qquad \forall \phi \in \mathcal{Q}^{(m,n)}_N(\Lambda). \tag{2.4}$$

As was shown in [19] that

$$P^{\mu,0}_{N,m,n} v = P_{N,m,n} v, \qquad \forall v \in H^{\mu}_{0,m,n,A}(\Lambda). \tag{2.5}$$

For inhomogeneous Dirichlet, Neumann and mixed boundary conditions, as well as for spectral element methods of high order equations in which the numerical solutions and some of their derivatives should match properly on all common boundaries of adjacent subdomains, we need certain quasi-orthogonal approximations. For this purpose, we introduce the following polynomials of degree $m+n-1$,

$$q^-_{m,n,j}(x) = \frac{1}{2^m j!}(1-x)^m \sum_{l=0}^{n-1-j} \frac{(m+l-1)!}{2^l l!(m-1)!}(1+x)^{l+j},$$

$$q^+_{m,n,j}(x) = \frac{(-1)^j}{2^n j!}(1+x)^n \sum_{l=0}^{m-1-j} \frac{(n+l-1)!}{2^l l!(n-1)!}(1-x)^{l+j}.$$

For any $v \in H^r_{m,n,A}(\Lambda)$ and $r \geq \max(m,n)$, we set

$$v_{m,n,b}(x) = \sum_{j=0}^{n-1} \partial_x^j v(-1) q^-_{m,n,j}(x) + \sum_{j=0}^{m-1} \partial_x^j v(1) q^+_{m,n,j}(x).$$

Furthermore, let $\widetilde{v}(x) = v(x) - v_{m,n,b}(x)$. The Jacobi quasi-orthogonal projection is given by

$$P^{\mu}_{N,m,n} v(x) = P^{\mu,0}_{N,m,n} \widetilde{v}(x) + v_{m,n,b}(x) \in \mathcal{P}_N(\Lambda). \tag{2.6}$$

We have (cf. [19])

$$\begin{aligned} &\partial_x^k P^{\mu}_{N,m,n} v(-1) = \partial_x^k v(-1), \qquad &&\text{for} \quad 0 \leq k \leq n-1, \\ &\partial_x^k P^{\mu}_{N,m,n} v(1) = \partial_x^k v(1), \qquad &&\text{for} \quad 0 \leq k \leq m-1. \end{aligned} \tag{2.7}$$

Throughout this paper, we denote by $c$ a generic positive constant independent of any function and $N$. According to (2.20) of [19], we know that if $v \in H^{\max(m,n)}_{m,n,A}(\Lambda)$, $\partial_x^r v \in$

$L^2_{\chi^{(-m+r,-n+r)}}(\Lambda)$, integers $m,n,r\geq 1$, $N\geq m+n$, $0\leq k\leq r\leq N+1$ and $\max(m,n,k)\leq \mu\leq m+n$, then

$$\begin{aligned}&||\partial_x^k(P^{\mu}_{N,m,n}v-v)||_{\chi^{(-m+k,-n+k)}}\\ \leq\ & cN^{k-r}(||\partial_x^r v||_{\chi^{(-m+r,-n+r)}}+||\partial_x^r v_{m,n,b}||_{\chi^{(-m+r,-n+r)}}).\end{aligned} \tag{2.8}$$

In particular, if $r\geq m+n$ (or $\max(m,n)\leq r$ and $1\leq m,n\leq 4$), then

$$||\partial_x^k(P^{\mu}_{N,m,n}v-v)||_{\chi^{(-m+k,-n+k)}}\leq cN^{k-r}||\partial_x^r v||_{\chi^{(-m+r,-n+r)}}. \tag{2.9}$$

The projection $P^{\mu}_{N,\sigma,\lambda}v$ possesses an interesting property, playing an important role in the spectral method of high and odd order problems. In fact (cf. [19]), for any $\phi\in\mathcal{Q}^{(\lambda,\lambda)}_{N-\sigma+\lambda}(\Lambda)$ and $\max(\sigma,\lambda)\leq\mu\leq m+n$,

$$\Big(\partial_x^{\sigma}(P^{\mu}_{N,\sigma,\lambda}v-v),\partial_x^{\lambda}\phi\Big)=0. \tag{2.10}$$

## 3. Spectral Element Method for Even Order Problems

### 3.1. Spectral element scheme.

The generalized Jacobi orthogonal approximation was applied successfully to numerical solutions of high order problems with homogeneous Dirichlet boundary conditions, see [17,18]. But, it seems not powerful enough for dealing with mixed inhomogeneous boundary value problems of high order. Especially, it is not easy to design proper spectral element method for high order problems. In this section, we design a new spectral element method for such problems.

As an example, we consider the following $2m$ order problem with mixed inhomogeneous Dirichlet-Neumann boundary conditions,

$$\begin{cases}(-1)^m\partial_x^{2m}U(x)+U(x)=f(x), & x\in I=(a,b),\\ \partial_x^m U(a)=\alpha_m,\quad \partial_x^m U(b)=\beta_m, & \\ \partial_x^k U(a)=\alpha_k,\quad \partial_x^k U(b)=\beta_k, & 0\leq k\leq m-2.\end{cases} \tag{3.1}$$

Let

$$\begin{aligned}&V(I)=\Big\{v\in H^m(I)\mid \partial_x^k v(a)=\alpha_k,\ \partial_x^k v(b)=\beta_k,\ 0\leq k\leq m-2\Big\},\\ &\bar{V}(I)=H^m(I)\cap H_0^{m-1}(I).\end{aligned}$$

The weak formulation of (3.1) is to find $U\in V(I)$ such that

$$\begin{aligned}&(\partial_x^m U,\partial_x^m v)+(U,v)+(-1)^{m-1}\beta_m\partial_x^{m-1}v(b)+(-1)^m\alpha_m\partial_x^{m-1}v(a)\\ =&(f,v),\quad \forall v\in\bar{V}(I).\end{aligned} \tag{3.2}$$

If $f\in V'(I)$, then (3.2) admits a unique solution.

Let $M$ be a positive integer, and $a=x_0<x_1<\cdots<x_M=b$. Denote $I_i=(x_{i-1},\ x_i)$ and $h_i=x_i-x_{i-1}$. For integer $N_i>0$, $\mathcal{P}_{N_i}(I_i)$ stands for the set of polynomials defined on $I_i$ of degree at most $N_i$. Let $\mathbf{N}=(N_1,N_2,\cdots,N_M)$, and

$$V_{\mathbf{N}}(I)=\Big\{\phi\in V(I)\mid \phi|_{I_i}\in\mathcal{P}_{N_i}(I_i)\ \text{ for } 1\leq i\leq M\Big\},$$

$$\bar{V}_{\mathbf{N}}(I)=\Big\{\phi\in\bar{V}(I)\mid\phi|_{I_i}\in\mathcal{P}_{N_i}(I_i)\quad\text{ for }\quad 1\le i\le M\Big\}.$$

The spectral element scheme for solving (3.2) is to seek $u_{\mathbf{N}}\in V_{\mathbf{N}}(I)$ such that

$$\begin{aligned}&(\partial_x^m u_{\mathbf{N}},\partial_x^m\phi)+(u_{\mathbf{N}},\phi)+(-1)^{m-1}\beta_m\partial_x^{m-1}\phi(b)+(-1)^m\alpha_m\partial_x^{m-1}\phi(a)\\=&(f,\phi),\quad\forall\phi\in\bar{V}_{\mathbf{N}}(I).\end{aligned}\tag{3.3}$$

For error estimation of numerical solution, we need some preparations. To do this, let $m_i,n_i\ge 1$, and

$$\chi_i^{(m_i,n_i)}(x)=2^{m_i+n_i}h_i^{-m_i-n_i}(x_i-x)^{m_i}(x-x_{i-1})^{n_i},\qquad x\in I_i,\ 1\le i\le M.$$

We introduce the spaces

$$\begin{aligned}&H^{r_i}_{m_i,n_i,A}(I_i)=\Big\{v\mid v\text{ is measurable on }I_i,\text{ and }||v||_{H^{r_i}_{m_i,n_i,A}(I_i)}<\infty\Big\},\\&H^{r_i}_{0,m_i,n_i,A}(I_i)=\Big\{v\in H^{r_i}_{m_i,n_i,A}(I_i)\mid\partial_x^k v(x_{i-1})=0\quad\text{for}\quad 0\le k\le n_i-1,\\&\qquad\qquad\text{and}\quad\partial_x^k v(x_i)=0\quad\text{for}\quad 0\le k\le m_i-1\Big\},\end{aligned}$$

equipped with the following semi-norm and norm,

$$|v|_{H^{r_i}_{m_i,n_i,A}(I_i)}=||\partial_x^{r_i}v||_{L^2_{\chi_i^{(-m_i+r_i,-n_i+r_i)}}(I_i)},\qquad ||v||_{H^{r_i}_{m_i,n_i,A}(I_i)}=\left(\sum_{k=0}^{r_i}|v|^2_{H^k_{m_i,n_i,A}(I_i)}\right)^{\frac12}.$$

Next, we set

$$\begin{aligned}&Y_{i,l}^{(m_i,n_i)}(x)=\chi_i^{(m_i,n_i)}(x)J_{l-m_i-n_i}^{(m_i,n_i)}\left(\frac{2x-x_{i-1}-x_i}{h_i}\right),\qquad x\in I_i,\quad l\ge m_i+n_i,\\&\mathcal{Q}_{N_i}^{(m_i,n_i)}(I_i)=\operatorname{span}\Big\{Y_{i,l}^{(m_i,n_i)}(x),\ m_i+n_i\le l\le N_i\Big\}.\end{aligned}$$

Let $\hat{v}(\hat{x})=v(x)$ and $x=\frac12(h_i\hat{x}+x_{i-1}+x_i)$. The projection $P^{\mu_i,0}_{N_i,m_i,n_i}$ is similar to the projection $P^{\mu,0}_{N,m,n}$ given by (2.4). For integers $\max(m_i,n_i)\le\mu_i\le m_i+n_i$, we define the local projections $P^{\mu_i,0}_{N_i,m_i,n_i,i}:H^{\mu_i}_{0,m_i,n_i,A}(I_i)\to\mathcal{Q}^{(m_i,n_i)}_{N_i}(I_i)$ as

$$P^{\mu_i,0}_{N_i,m_i,n_i,i}v(x)=P^{\mu_i,0}_{N_i,m_i,n_i}\hat{v}(\hat{x})\Big|_{\hat{x}=\frac{2x-x_{i-1}-x_i}{h_i}},\qquad 1\le i\le M.$$

We also introduce the following functions: for $\le i\le M$,

$$\begin{aligned}&q^{i,\pm}_{m_i,n_i,j}(x)=q^{\pm}_{m_i,n_i,j}\left(\tfrac{2x-x_{i-1}-x_i}{h_i}\right),\\&v_{m_i,n_i,b_i}(x)=\sum_{j=0}^{n_i-1}\left(\frac{h_i}{2}\right)^j\partial_x^j v(x_{i-1})q^{i,-}_{m_i,n_i,j}(x)+\sum_{j=0}^{m_i-1}\left(\frac{h_i}{2}\right)^j\partial_x^j v(x_i)q^{i,+}_{m_i,n_i,j}(x),\end{aligned}$$

where $\partial_x^k v(x_0)=\alpha_k$ and $\partial_x^k v(x_M)=\beta_k,\ 0\le k\le m-2.$

The local quasi-orthogonal projections are defined by

$$P^{\mu_i}_{N_i,m_i,n_i,i}v(x)=P^{\mu_i,0}_{N_i,m_i,n_i,i}\left(v(x)-v_{m_i,n_i,b_i}(x)\right)+v_{m_i,n_i,b_i}(x),\qquad x\in I_i,\ 1\le i\le M.$$

If $v \in H^{\mu_i}_{m_i,n_i,A}(I_i)$, $\partial_x^{r_i} v \in L^2_{\chi_i^{(-m_i+r_i,-n_i+r_i)}}(I_i)$, integers $m_i, n_i, r_i \geq 1$, $N_i \geq m_i + n_i$,

$$\max(m_i, n_i) \leq \mu_i \leq m_i + n_i, \quad 0 \leq k \leq r_i \leq N_i + 1,$$

and $r_i > m_i + n_i$ (or $\max(m_i, n_i) \leq r_i$ and $m_i, n_i \leq 4$), then (2.9) leads to

$$||\partial_x^k (v - P^{\mu_i}_{N_i,m_i,n_i,i} v)||_{L^2_{\chi_i^{(-m_i+k,-n_i+k)}}(I_i)} \leq c h_i^{r_i-k} N_i^{k-r_i} ||\partial_x^{r_i} v||_{L^2_{\chi_i^{(-m_i+r_i,-n_i+r_i)}}(I_i)}. \tag{3.4}$$

We next construct a specific composite quasi-orthogonal projection defined on the whole interval $I$. Let $\mathbf{r} = (r_1, r_2, \cdots, r_M)$ and $\mathbf{h} = (h_1, h_2, \cdots, h_M)$. The projection $P^*_{\mathbf{N}} v(x)$ is defined by

$$P^*_{\mathbf{N}} v(x)|_{I_i} = P^m_{N_i,m,m,i} v(x), \qquad 1 \leq i \leq M.$$

For notational convenience, we introduce the following quantities,

$$\mathbb{A}(v,k) = \left( \sum_{i=1}^{M} ||\partial_x^k v||^2_{L^2_{\chi_i^{(-m+k,-m+k)}}(I_i)} \right)^{\frac{1}{2}},$$

$$\mathbb{B}(v,\mathbf{N},\mathbf{h},\mathbf{r},k) = \left( \sum_{i=1}^{M} h_i^{2r_i-2k} N_i^{2k-2r_i} ||\partial_x^{r_i} v||^2_{L^2_{\chi_i^{(-m+r_i,-m+r_i)}}(I_i)} \right)^{\frac{1}{2}}.$$

Assume that $m \leq r_i \leq N_i + 1$ and $N_i \geq 2m$ for $1 \leq i \leq M$. Then, by virtue of (3.4),

$$\mathbb{A}(v - P^*_{\mathbf{N}} v, k) \leq c \mathbb{B}(v, \mathbf{N}, \mathbf{h}, \mathbf{r}, k), \qquad 0 \leq k \leq m. \tag{3.5}$$

We now deal with the convergence of scheme (3.3). As usual, we introduce the auxiliary projection $\bar{P}^m_{\mathbf{N}} : V(I) \to V_{\mathbf{N}}(I)$, defined by

$$(\partial_x^2 (\bar{P}^m_{\mathbf{N}} v - v), \partial_x^m \phi) + (\bar{P}^m_{\mathbf{N}} v - v, \phi) = 0, \qquad \forall \phi \in \bar{V}_{\mathbf{N}}(I). \tag{3.6}$$

The above projection has a nice property. In fact, for any $v \in V(I)$ and $w \in V_{\mathbf{N}}(I)$, we have $\bar{P}^m_{\mathbf{N}} v - w \in \bar{V}_{\mathbf{N}}(I)$. Thus, we use (3.6) to deduce that

$$\begin{aligned}
&(\partial_x^m (v - w), \partial_x^m (v - w)) + (v - w, v - w) \\
&\quad = (\partial_x^m (v - \bar{P}^m_{\mathbf{N}} v), \partial_x^m (v - \bar{P}^m_{\mathbf{N}} v)) + (v - \bar{P}^m_{\mathbf{N}} v, v - \bar{P}^m_{\mathbf{N}} v) \\
&\qquad + (\partial_x^m (\bar{P}^m_{\mathbf{N}} v - w), \partial_x^m (\bar{P}^m_{\mathbf{N}} v - w)) + (\bar{P}^m_{\mathbf{N}} v - w, \bar{P}^m_{\mathbf{N}} v - w) \\
&\qquad + 2(\partial_x^m (v - \bar{P}^m_{\mathbf{N}} v), \partial_x^m (\bar{P}^m_{\mathbf{N}} v - w)) + 2(v - \bar{P}^m_{\mathbf{N}} v, \bar{P}^m_{\mathbf{N}} v - w) \\
&\quad \geq (\partial_x^m (v - \bar{P}^m_{\mathbf{N}} v), \partial_x^m (v - \bar{P}^m_{\mathbf{N}} v)) + (v - \bar{P}^m_{\mathbf{N}} v, v - \bar{P}^m_{\mathbf{N}} v).
\end{aligned} \tag{3.7}$$

Furthermore, since $V_{\mathbf{N}}(I) \subset V(I)$ and $\bar{V}_{\mathbf{N}}(I) \subset \bar{V}(I)$, we use (3.2) to obtain

$$\begin{aligned}
&(\partial_x^m \bar{P}^m_{\mathbf{N}} U, \partial_x^m \phi) + (\bar{P}^m_{\mathbf{N}} U, \phi) + (-1)^{m-1} \beta_m \partial_x^{m-1} \phi(b) \\
&\quad + (-1)^m \alpha_m \partial_x^{m-1} \phi(a) = (f, \phi), \qquad \forall \phi \in \bar{V}_{\mathbf{N}}(I).
\end{aligned}$$

Let $\tilde{U}_{\mathbf{N}} = u_{\mathbf{N}} - \bar{P}^m_{\mathbf{N}} U$. Subtracting the above equation from (3.3) yields

$$(\partial_x^m \tilde{U}_{\mathbf{N}}, \partial_x^m \phi) + (\tilde{U}_{\mathbf{N}}, \phi) = 0, \qquad \forall \phi \in \bar{V}_{\mathbf{N}}(I). \tag{3.8}$$

Taking $\phi = \tilde{U}_{\mathbf{N}}$ in (3.8), we obtain $\tilde{U}_{\mathbf{N}}(x) \equiv 0$, i.e., $u_{\mathbf{N}} = \bar{P}^m_{\mathbf{N}} U$. Moreover, by the definition of $P^*_{\mathbf{N}} v(x)$, we have $P^*_{\mathbf{N}} U \in V_{\mathbf{N}}(I)$. Therefore, we use the Gagliardo-Nirenberg inequality, the property of projection $\bar{P}^m_{\mathbf{N}}U$, (3.7) with $v = U$ and $w = P^*_{\mathbf{N}}U$, and (3.5) successively, to derive that

$$\begin{aligned}\|U - u_{\mathbf{N}}\|_{H^m(I)} &\le c\Big(\|\partial_x^m(U - u_{\mathbf{N}})\|_{L^2(I)} + \|U - u_{\mathbf{N}}\|_{L^2(I)}\Big)\\ &= c\Big(\|\partial_x^m(U - \bar{P}^m_{\mathbf{N}}U)\|_{L^2(I)} + \|U - \bar{P}^m_{\mathbf{N}}U\|_{L^2(I)}\Big)\\ &\le c\Big(\|\partial_x^m(U - P^*_{\mathbf{N}}U)\|_{L^2(I)} + \|U - P^*_{\mathbf{N}}U\|_{L^2(I)}\Big)\\ &\le c\Big(\mathbb{A}(U - P^*_{\mathbf{N}}U, m) + \mathbb{A}(U - P^*_{\mathbf{N}}U, 0)\Big)\\ &\le c\mathbb{B}(U, \mathbf{N}, \mathbf{h}, \mathbf{r}, m). \end{aligned} \tag{3.9}$$

If $h_i = h, N_i = N \ge 2m$ and $m \le r_i \le N + 1$ for $1 \le i \le M$, then

$$\|U - u_{\mathbf{N}}\|_{H^m(I)} \le c\left(\sum_{i=1}^{M}(\frac{h}{N})^{2r_i-2m}||\partial_x^{r_i}U||^2_{L^2_{\chi_i^{(-m+r_i,-m+r_i)}}(I_i)}\right)^{\frac{1}{2}}.$$

As we know, it is not easy to derive the optimal estimate of $\|U - u_{\mathbf{N}}\|$ for the mixed boundary value problems. We now use a unusual duality argument to derive it. For simplicity of statements, we focus on the homogeneous boundary conditions, i.e., $\alpha_m = \beta_m = \alpha_k = \beta_k = 0$ for $0 \le k \le m-2$. In this case, $V(I) = \bar{V}(I) = H^m(I) \cap H_0^{m-1}(I)$ and $V_{\mathbf{N}}(I) = \bar{V}_{\mathbf{N}}(I)$. Let

$$W(I) = \Big\{v \in V(I) \mid \partial_x^m v(a) = \partial_x^m v(b) = 0\Big\}.$$

For $g \in L^2(I)$, we consider an auxiliary problem. It is to find $w \in W(I)$ such that

$$(\partial_x^m w, \partial_x^m z) + (w, z) = (g, z), \qquad \forall z \in V(I). \tag{3.10}$$

Taking $z = U - u_{\mathbf{N}} \in V(I)$ in (3.10), we obtain

$$\Big(\partial_x^m w, \partial_x^m(U - u_{\mathbf{N}})\Big) + (w, U - u_{\mathbf{N}}) = (g, U - u_{\mathbf{N}}).$$

On the other hand, as it was shown before, $u_{\mathbf{N}} = \bar{P}^m_{\mathbf{N}}U$. Moreover, $\bar{P}^m_{\mathbf{N}}w \in \bar{V}_{\mathbf{N}}(I)$. As a result, we have from (3.6) with $v = U$ and $\phi = \bar{P}^m_{\mathbf{N}}w$ to deduce that

$$\Big(\partial_x^m \bar{P}^m_{\mathbf{N}}w, \partial_x^m(U - u_{\mathbf{N}})\Big) + (\bar{P}^m_{\mathbf{N}}w, U - u_{\mathbf{N}}) = 0.$$

A combination of the above two equalities leads to

$$\Big(\partial_x^m(w - \bar{P}^m_{\mathbf{N}}w), \partial_x^m(U - u_{\mathbf{N}})\Big) + (w - \bar{P}^m_{\mathbf{N}}w, U - u_{\mathbf{N}}) = (g, U - u_{\mathbf{N}}).$$

Next, let $\tilde{\mathbf{r}} = (2m, 2m, \cdots, 2m)$. Then, it follows from (3.9) that

$$|(g, U - u_{\mathbf{N}})| \le c\mathbb{B}(U, \mathbf{N}, \mathbf{h}, \mathbf{r}, m)\mathbb{B}(w, \mathbf{N}, \mathbf{h}, \tilde{\mathbf{r}}, m). \tag{3.11}$$

It can be checked that

$$\mathbb{B}(w, \mathbf{N}, \mathbf{h}, \tilde{\mathbf{r}}, m) \le c\left(\sum_{i=1}^{M}(\frac{h_i}{N_i})^{2m}||\partial_x^{2m}w||^2_{L^2(I_i)}\right)^{\frac{1}{2}}. \tag{3.12}$$

Furthermore, we derive from (3.10) that in sense of distribution,

$$(-1)^m \partial_x^{2m} w(x) + w(x) = g(x).$$

In addition, $\partial_x^m w(a) = \partial_x^m w(b) = 0$. Multiplying the above equation by $w(x)$ and using integration by parts, we obtain that $\|w\| \le \|g\|$. Accordingly,

$$\|\partial_x^{2m} w\|^2 \le 2\|g\|^2 + 2\|w\|^2 \le 4\|g\|^2.$$

Finally, we use the above inequality, along with (3.11) and (3.12), to deduce that for the uniform mode $N_i = N$ and the uniform step size $h_i = h$,

$$\|U - u_{\mathbf{N}}\| = \sup_{g \in L^2(I), g \ne 0} \frac{|(U - u_{\mathbf{N}}, g)|}{\|g\|} \le c \left(\frac{h}{N}\right)^m \mathbb{B}(U, \mathbf{N}, \mathbf{h}, \mathbf{r}, m),$$

which is the optimal error estimate.

### 3.2. Spectral element scheme with essential imposition of boundary conditions.

We now turn to the spectral element scheme for (3.1), with essential imposition of boundary conditions. To do this, let

$$\widetilde{V}_{\mathbf{N}}(I) = \Big\{ \phi \in V(I) \mid \phi|_{I_i} \in \mathcal{P}_{N_i}(I_i) \ \text{ for } 1 \le i \le M, \text{ and } \partial_x^m \phi(a) = \alpha_m, \partial_x^m \phi(b) = \beta_m \Big\},$$
$$\bar{\widetilde{V}}_{\mathbf{N}}(I) = \Big\{ \phi \in \bar{V}(I) \mid \phi|_{I_i} \in \mathcal{P}_{N_i}(I_i) \ \text{ for } 1 \le i \le M, \text{ and } \partial_x^m \phi(a) = \partial_x^m \phi(b) = 0 \Big\}.$$

The spectral element scheme for (3.2) is to seek $u_{\mathbf{N}} \in \widetilde{V}_{\mathbf{N}}(I)$ such that

$$\begin{aligned} &(\partial_x^m u_{\mathbf{N}}, \partial_x^m \phi) + (u_{\mathbf{N}}, \phi) + (-1)^{m-1} \beta_m \partial_x^{m-1} \phi(b) + (-1)^m \alpha_m \partial_x^{m-1} \phi(a) \\ &= (f, \phi), \quad \forall \phi \in \bar{\widetilde{V}}_{\mathbf{N}}(I). \end{aligned} \tag{3.13}$$

The new composite quasi-orthogonal projection $P_{\mathbf{N}}^* v(x)$ is defined by

$$P_{\mathbf{N}}^* v(x)|_{I_i} = \begin{cases} P_{N_1,m,m+1,1}^{m+1} v(x), & \\ P_{N_i,m,m,i}^{m} v(x), & 2 \le i \le M-1, \\ P_{N_M,m+1,m,M}^{m+1} v(x). & \end{cases}$$

We introduce the following quantities,

$$\begin{aligned} \widetilde{\mathbb{A}}(v,k) = \Bigg( &\sum_{i=2}^{M-1} ||\partial_x^k v||^2_{L^2_{\chi_i^{(-m+k,-m+k)}}(I_i)} + ||\partial_x^k v||^2_{L^2_{\chi_1^{(-m+k,-m-1+k)}}(I_1)} \\ &+ ||\partial_x^k v||^2_{L^2_{\chi_M^{(-m-1+k,-m+k)}}(I_M)} \Bigg)^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} \widetilde{\mathbb{B}}(v, \mathbf{N}, \mathbf{h}, \mathbf{r}, k) = \Bigg( &\sum_{i=2}^{M-1} h_i^{2r_i - 2k} N_i^{2k - 2r_i} ||\partial_x^{r_i} v||^2_{L^2_{\chi_i^{(-m+r_i,-m+r_i)}}(I_i)} \\ &+ h_1^{2r_1 - 2k} N_1^{2k - 2r_1} ||\partial_x^{r_1} v||^2_{L^2_{\chi_1^{(-m+r_1,-m-1+r_1)}}(I_1)} \\ &+ h_M^{2r_M - 2k} N_M^{2k - 2r_M} ||\partial_x^{r_M} v||^2_{L^2_{\chi_M^{(-m-1+r_M,-m+r_M)}}(I_M)} \Bigg)^{\frac{1}{2}}. \end{aligned}$$

Assume that $m \le r_i \le N_i + 1$ and $N_i \ge 2m$ for $2 \le i \le M-1$, while $m+1 \le r_i \le N_i + 1$ and $N_i \ge 2m+1$ for $i = 1, M$. Then, by virtue of (3.4),

$$\widetilde{\mathbb{A}}(v - P_{\mathbf{N}}^* v, k) \le c\widetilde{\mathbb{B}}(v, \mathbf{N}, \mathbf{h}, \mathbf{r}, k), \qquad 0 \le k \le m. \tag{3.14}$$

We now deal with the convergence of scheme (3.13). We introduce the auxiliary projection $\widetilde{P}_{\mathbf{N}}^m : V(I) \to \widetilde{V}_{\mathbf{N}}(I)$, defined by

$$\Big(\partial_x^m(\widetilde{P}_{\mathbf{N}}^m v - v), \partial_x^m \phi\Big) + (\widetilde{P}_{\mathbf{N}}^m v - v, \phi) = 0, \qquad \forall \phi \in \bar{\widetilde{V}}_{\mathbf{N}}(I). \tag{3.15}$$

For any $v \in V(I)$ and $w \in \widetilde{V}_{\mathbf{N}}(I)$, we have $\widetilde{P}_{\mathbf{N}}^m v - w \in \bar{\widetilde{V}}_{\mathbf{N}}(I)$. Thus, with the aid of (3.15), an argument similar to derivation of (3.7) leads to

$$\begin{aligned}&\Big(\partial_x^m(v-w), \partial_x^m(v-w)\Big) + (v-w, v-w)\\ \ge &\Big(\partial_x^m(v - \widetilde{P}_{\mathbf{N}}^m v), \partial_x^m(v - \widetilde{P}_{\mathbf{N}}^m v)\Big) + (v - \widetilde{P}_{\mathbf{N}}^m v, v - \widetilde{P}_{\mathbf{N}}^m v).\end{aligned} \tag{3.16}$$

Moreover, since $\bar{\widetilde{V}}_{\mathbf{N}}(I) \subset \bar{V}(I)$, we use (3.2) to obtain

$$\begin{aligned}&(\partial_x^m \widetilde{P}_{\mathbf{N}}^m U, \partial_x^m \phi) + (\widetilde{P}_{\mathbf{N}}^m U, \phi) + (-1)^{m-1}\beta_m \partial_x^{m-1}\phi(b)\\ &\quad + (-1)^m \alpha_m \partial_x^{m-1}\phi(a) = (f, \phi), \qquad \forall \phi \in \bar{\widetilde{V}}_{\mathbf{N}}(I).\end{aligned}$$

Let $\tilde{U}_{\mathbf{N}} = u_{\mathbf{N}} - \widetilde{P}_{\mathbf{N}}^m U$. Subtracting the above equation from (3.13) yields

$$(\partial_x^m \tilde{U}_{\mathbf{N}}, \partial_x^m \phi) + (\tilde{U}_{\mathbf{N}}, \phi) = 0, \qquad \forall \phi \in \bar{\widetilde{V}}_{\mathbf{N}}(I). \tag{3.17}$$

Taking $\phi = \tilde{U}_{\mathbf{N}}$ in (3.17), we obtain $\tilde{U}_{\mathbf{N}}(x) \equiv 0$, i.e., $u_{\mathbf{N}} = \widetilde{P}_{\mathbf{N}}^m U$. Moreover, by the definition of $P_{\mathbf{N}}^* v(x)$, we have $P_{\mathbf{N}}^* U \in \widetilde{V}_{\mathbf{N}}(I)$. Therefore, we use the Gagliardo-Nirenberg inequality, the property of projection $\widetilde{P}_{\mathbf{N}}^m U$, (3.16) with $v = U$ and $w = P_{\mathbf{N}}^* U$, and (3.14) successively, to derive that

$$\begin{aligned}\|U - u_{\mathbf{N}}\|_{H^m(I)} &\le c\Big(\|\partial_x^m(U - u_{\mathbf{N}})\|_{L^2(I)} + \|U - u_{\mathbf{N}}\|_{L^2(I)}\Big)\\ &= c\Big(\|\partial_x^m(U - \widetilde{P}_{\mathbf{N}}^m U)\|_{L^2(I)} + \|U - \widetilde{P}_{\mathbf{N}}^m U\|_{L^2(I)}\Big)\\ &\le c\Big(\|\partial_x^m(U - P_{\mathbf{N}}^* U)\|_{L^2(I)} + \|U - P_{\mathbf{N}}^* U\|_{L^2(I)}\Big)\\ &\le c\Big(\widetilde{\mathbb{A}}(U - P_{\mathbf{N}}^* U, m) + \widetilde{\mathbb{A}}(U - P_{\mathbf{N}}^* U, 0)\Big)\\ &\le c\widetilde{\mathbb{B}}(U, \mathbf{N}, \mathbf{h}, \mathbf{r}, m).\end{aligned} \tag{3.18}$$

If $h_i = h, N_i = N \ge 2m+1$, $m \le r_i \le N+1$ for $2 \le i \le M-1$, and $m+1 \le r_1, r_M \le N+1$, then

$$\begin{aligned}\|U - u_{\mathbf{N}}\|_{H^m(I)} \le c\Bigg(&\sum_{i=2}^{M-1}\left(\frac{h}{N}\right)^{2r_i - 2m} ||\partial_x^{r_i} U||^2_{L^2_{\chi_i^{(-m+r_i, -m+r_i)}}(I_i)}\\ &+ \left(\frac{h}{N}\right)^{2r_1 - 2m} ||\partial_x^{r_1} U||^2_{L^2_{\chi_1^{(-m+r_1, -m-1+r_1)}}(I_1)}\\ &+ \left(\frac{h}{N}\right)^{2r_M - 2m} ||\partial_x^{r_M} U||^2_{L^2_{\chi_M^{(-m-1+r_M, -m+r_M)}}(I_M)}\Bigg)^{\frac{1}{2}}.\end{aligned}$$

We now derive the optimal error estimate of $\|U - u_{\mathbf{N}}\|$, for the case with the homogeneous boundary conditions. In this case, $V(I) = \bar{V}(I) = H^m(I) \cap H_0^{m-1}(I)$ and $\widetilde{V}_{\mathbf{N}}(I) = \bar{\bar{V}}_{\mathbf{N}}(I)$. Let $W(I)$ be the same as before, and $g \in L^2(I)$. We consider an auxiliary problem. It is to find $w \in W(I)$ such that

$$\Big(\partial_x^m w, \partial_x^m z\Big) + (w, z) = (g, z), \qquad \forall z \in W(I). \tag{3.19}$$

Taking $z = U - u_{\mathbf{N}} \in W(I)$ in (3.19), we obtain

$$\Big(\partial_x^m w, \partial_x^m (U - u_{\mathbf{N}})\Big) + (w, U - u_{\mathbf{N}}) = (g, U - u_{\mathbf{N}}).$$

Since $u_{\mathbf{N}} = \widetilde{P}_{\mathbf{N}}^m U$ and $\widetilde{P}_{\mathbf{N}}^m w \in \bar{\bar{V}}_{\mathbf{N}}(I)$, we use (3.15) with $v = U$ and $\phi = \widetilde{P}_{\mathbf{N}}^m w$ to obtain

$$\Big(\partial_x^m \widetilde{P}_{\mathbf{N}}^m w, \partial_x^m (U - u_{\mathbf{N}})\Big) + (\widetilde{P}_{\mathbf{N}}^m w, U - u_{\mathbf{N}}) = 0.$$

A combination of the above two equalities leads to

$$\Big(\partial_x^m (w - \widetilde{P}_{\mathbf{N}}^m w),\ \partial_x^m (U - u_{\mathbf{N}})\Big) + (w - \widetilde{P}_{\mathbf{N}}^m w, U - u_{\mathbf{N}}) = (g, U - u_{\mathbf{N}}).$$

Next, let $\tilde{\mathbf{r}} = (2m, 2m, \cdots, 2m)$. Then, by virtue of (3.18),

$$|(g, U - u_{\mathbf{N}})| \le c\widetilde{\mathbb{B}}(U, \mathbf{N}, \mathbf{h}, \mathbf{r}, m)\widetilde{\mathbb{B}}(w, \mathbf{N}, \mathbf{h}, \tilde{\mathbf{r}}, m). \tag{3.20}$$

It can be checked that

$$\widetilde{\mathbb{B}}(w, \mathbf{N}, \mathbf{h}, \tilde{\mathbf{r}}, m) \le c\left(\sum_{i=1}^{M} \left(\frac{h_i}{N_i}\right)^{2m} ||\partial_x^{2m} w||^2_{L^2(I_i)}\right)^{\frac{1}{2}}. \tag{3.21}$$

Furthermore, we derive from (3.19) that in sense of distribution,

$$(-1)^m \partial_x^{2m} w(x) + w(x) = g(x),$$

with $\partial_x^m w(a) = \partial_x^m w(b) = 0$. Multiplying the above equation by $w(x)$ and using integration by parts, we obtain $\|w\| \le \|g\|$. Accordingly,

$$\|\partial_x^{2m} w\|^2 \le 2\|g\|^2 + 2\|w\|^2 \le 4\|g\|^2.$$

Finally, we use the above inequality, along with (3.20) and (3.21), to deduce that for the uniform mode $N_i = N$ and the uniform step size $h_i = h$,

$$\|U - u_{\mathbf{N}}\| = \sup_{g \in L^2(I), g \neq 0} \frac{|(U - u_{\mathbf{N}}, g)|}{\|g\|} \le c\left(\frac{h}{N}\right)^m \widetilde{\mathbb{B}}(U, \mathbf{N}, \mathbf{h}, \mathbf{r}, m), \tag{3.22}$$

which is the optimal error estimate.

**Remark 3.1.** Auteri et al. [1] proposed a Legendre spectral method for second order elliptic equations, with essential imposition of Neumann conditions. Doha and Bhrawy [9] considered an efficient direct solver for multidimensional elliptic Robin boundary value problems using the Legendre spectral method. But they did not adopt the partition of considered domains. We also refer to the work of [10,29,30]. In this paper, we developed a new and simpler spectral element method suitable for high order problems, with the essential imposition of mixed Dirichlet-Neumann boundary conditions.

## 4. Spectral Method for High and Odd Order Problems

We now turn to the spectral method for high and odd order problems. For simplicity of statements, we consider the following problem with homogeneous boundary condition (cf. Ma and Sun [26] for $m = 1$),

$$\begin{cases} \partial_t U(x,t) + (-1)^{m+1}\partial_x^{2m+1}U(x,t) = f(x,t), & x \in \Lambda,\ t > 0, \\ \partial_x^k U(\pm 1,t) = \partial_x^m U(1,t) = 0, & 0 \le k \le m-1,\ t > 0, \\ U(x,0) = U_0(x), & x \in \bar{\Lambda}. \end{cases} \tag{4.1}$$

Let $V(\Lambda) = \{v \mid v \in H^{m+1}(\Lambda) \cap H_0^m(\Lambda) \text{ and } \partial_x^m v(1) = 0\}$. The weak form of (4.1) is to find $U \in L^\infty(0,T;L^2(\Lambda)) \cap L^2(0,T;V(\Lambda))$, such that

$$\Big(\partial_t U(t), v\Big) - \Big(\partial_x^{m+1}U(t), \partial_x^m v\Big) = \Big(f(t), v\Big), \qquad \forall v \in H_0^m(\Lambda)\ \ , t > 0, \tag{4.2}$$

with $U(0) = U_0$. If $U_0 \in V(\Lambda)$ and $f \in L^2(0,T;H^{m+1}(\Lambda))$, then (4.2) has a unique solution.

Let $V_N(\Lambda) = V(\Lambda) \cap \mathcal{P}_N(\Lambda)$ and $V_{N-1}^*(\Lambda) = H_0^m(\Lambda) \cap \mathcal{P}_{N-1}(\Lambda)$. The spectral Petrov-Galerkin method for (4.2) is to seek $u_N(t) \in V_N(\Lambda)$, such that

$$\Big(\partial_t u_N(t), \phi\Big) - \Big(\partial_x^{m+1}u_N(t), \partial_x^m \phi\Big) = \Big(f(t), \phi\Big), \qquad \forall \phi \in V_{N-1}^*(\Lambda),\ t > 0, \tag{4.3}$$

with $u_N(0) = P_{N,m+1,m}^{m+1}U_0$. Due to (2.7), we have $u_N(0) \in V_N(\Lambda)$.

For the analysis of convergence, we set $U_N = P_{N,m+1,m}^{m+1}U \in V_N(\Lambda)$. By virtue of (2.10) with $\mu = \sigma = m+1$ and $\lambda = m$, we have from (4.2) that

$$\Big(\partial_t U_N(t), \phi\Big) - \Big(\partial_x^{m+1}U_N(t), \partial_x^m \phi\Big) + G(\phi,t) = \Big(f(t), \phi\Big), \quad \forall \phi \in V_{N-1}^*(\Lambda),\ t > 0, \tag{4.4}$$

with

$$G(\phi,t) = (\partial_t(U(t) - U_N(t)), \phi).$$

Further, let $\tilde{U}_N = u_N - U_N$. Subtracting (4.4) from (4.3) yields

$$\Big(\partial_t \tilde{U}_N(t), \phi\Big) - \Big(\partial_x^{m+1}\tilde{U}_N(t), \partial_x^m \phi\Big) = G(\phi,t), \qquad \forall \phi \in V_{N-1}^*(\Lambda),\ t > 0, \tag{4.5}$$

In addition, $\tilde{U}_N(0) = 0$.

Next, following the same line as in [25,26], we introduce the polynomial $\eta_N = (1-x)^{-1}\tilde{U}_N \in V_{N-1}^*(\Lambda)$. A direct calculation shows

$$\begin{aligned} &-\Big(\partial_x^{m+1}\tilde{U}_N(t), \partial_x^m \eta_N(t)\Big) \\ =& -\Big(\partial_x^{m+1}\eta_N(t), (1-x)\partial_x^m \eta_N(t)\Big) + (m+1)||\partial_x^m \eta_N(t)||^2 \\ =& (m+\frac{1}{2})||\partial_x^m \eta_N(t)||^2 + \Big(\partial_x^m \eta_N(-1,t)\Big)^2. \end{aligned} \tag{4.6}$$

Thanks to (2.9) and the Poincaré inequality, we deduce that

$$\begin{aligned} 2|G(\eta_N,t)| &\le 2||\partial_t(U_N(t) - U(t)||||\partial_x^m \eta_N(t)|| \\ &\le \frac{1}{2}||\partial_x^m \eta_N(t)||^2 + cN^{-2r}||\partial_t \partial_x^r U(t)||^2_{\chi^{(r-m-1,r-m)}}. \end{aligned} \tag{4.7}$$

By taking $\phi = 2\eta_N$ in (4.5) and inserting (4.6) and (4.7) into the resulting equation, we obtain

$$\partial_t ||\tilde{U}_N(t)||^2_{\chi^{(-1,0)}} + m||\partial_x^m \eta_N(t)||^2 + (\partial_x^m \eta_N(-1,t))^2 \le cN^{-2r}||\partial_t \partial_x^r U(t)||^2_{\chi^{(r-m-1,r-m)}}.$$

By integrating the above inequality and using (2.9) again, we conclude that for integer $r \ge m+1$,

$$\begin{aligned} ||U(t) - u_N(t)||^2_{\chi^{(-1,0)}} &\le 2||\tilde{U}_N(t)||^2_{\chi^{(-1,0)}} + 2||U(t) - U_N(t)||^2_{\chi^{(-1,0)}} \\ &\le cN^{-2r}\left(\int_0^t ||\partial_\xi \partial_x^r U(\xi)||^2_{\chi^{(r-m-1,r-m)}} d\xi + ||\partial_x^r U(t)||^2_{\chi^{(r-m-1,r-m)}}\right). \end{aligned} \tag{4.8}$$

**Remark 4.1.** Ma and Sun [25, 26] proposed a spectral method for the KdV-type equation, i.e., the problem (4.1) with $m = 1$. In the numerical analysis, they took a good comparison function $P_N^* v$, which leads to much better error estimation. In fact, $P_N^* v = P^2_{N,2,1} v$ for any $v \in H^2(\Lambda) \cap H_0^1(\Lambda)$ with $\partial_x v(1) = 0$.

**Remark 4.2.** Guo et al. [17] also considered a spectral method for steady problem corresponding to (4.1). By (3.18) of that paper, we has only

$$||U - u_N||^2_{\chi^{(-1,1)}} \le cN^{2m-2r-2}||\partial_x^r U||^2_{\chi^{(r-m-1,r-m)}}.$$

If, for instance, $m = 1$, then

$$||U - u_N||^2_{\chi^{(-1,1)}} \le cN^{-2r}||\partial_x^r U||^2_{\chi^{(r-2,r-1)}}.$$

Since the approximated function $U$ does not vanish at $x = 1$, this estimate could not ensure the high accuracy as $x \to 1$. Whereas, by virtue of (4.8) of this paper, for steady problem (4.1) with $m = 1$, we have

$$||U - u_N||^2_{\chi^{(-1,0)}} \le cN^{-2r}||\partial_x^r U||^2_{\chi^{(r-2,r-1)}}.$$

Clearly, our new result improves the corresponding result of [17] essentially. Especially, it ensure the high accuracy of numerical solution as $x \to 1$. It is noted that the result (4.8) with $m = 1$ was also obtained by Ma and Sun [25, 26].

## 5. Numerical Results

In this section, we present some numerical results. For fixedness and simplicity of statements, we focus on the case with $m = 2$ in (3.1).

We first describe the numerical implementation of scheme (3.3). In actual computation, we need five kinds of base functions. We first introduce the local polynomials corresponding to the subintervals $I_i$, such that for $1 \le i \le M$,

$$\psi_{i,l}(x) = \begin{cases} Y_{i,l}^{(2,2)}(x), & x \in \bar{I}_i, \\ 0, & \text{otherwise}. \end{cases} \tag{5.1}$$

The sets of $\psi_{i,l}(x)$ form the basis of trail function spaces $Q_{N_i}^{(2,2)}(I_i)$ for $1 \le i \le M$.

Next, we define the basis functions for matching approximated functions at the interior nodes $x_i$, such that for $1 \le i \le M - 1$,

$$\sigma_i(x) = \begin{cases} q_{2,2,0}^{i,+}(x), & x \in \bar{I}_i, \\ q_{2,2,0}^{i+1,-}(x), & x \in \bar{I}_{i+1}, \\ 0, & \text{otherwise}. \end{cases} \tag{5.2}$$

Obviously, $\sigma_i(x_i)=1$ and $\partial_x\sigma_i(x_i)=0$, $1\le i\le M-1$.

Similarly, we define the basis functions for matching the first order derivatives of underlying functions at the interior nodes $x_i$, such that for $1\le i\le M-1$,

$$\lambda_i(x)=\begin{cases}\frac{h_i}{2}q^{i,+}_{2,2,1}(x), & x\in\bar{I}_i,\\ \frac{h_{i+1}}{2}q^{i+1,-}_{2,2,1}(x), & x\in\bar{I}_{i+1},\\ 0, & \text{otherwise.}\end{cases} \tag{5.3}$$

Evidently, $\lambda_i(x_i)=0$ and $\partial_x\lambda_i(x_i)=1$, $1\le i\le M-1$.

Furthermore, we take the basis functions for matching approximated functions at the boundary ($x=x_0,x_M$), as

$$\zeta_1(x)=\begin{cases}q^{1,-}_{2,2,0}(x), & x\in\bar{I}_1,\\ 0, & \text{otherwise,}\end{cases} \tag{5.4}$$

$$\zeta_M(x)=\begin{cases}q^{M,+}_{2,2,0}(x), & x\in\bar{I}_M,\\ 0, & \text{otherwise.}\end{cases} \tag{5.5}$$

We also take the basis functions for matching the first order derivatives of underlying functions at the boundary ($x=x_0,x_M$), as

$$\xi_1(x)=\begin{cases}\frac{h_1}{2}q^{1,-}_{2,2,1}(x), & x\in\bar{I}_1,\\ 0, & \text{otherwise,}\end{cases} \tag{5.6}$$

$$\xi_M(x)=\begin{cases}\frac{h_M}{2}q^{M,+}_{2,2,1}(x), & x\in\bar{I}_M,\\ 0, & \text{otherwise.}\end{cases} \tag{5.7}$$

We denote by $b_i=u_{\mathbf{N}}(x_i)$ and $c_i=\partial_x u_{\mathbf{N}}(x_i)$ for $0\le i\le M$. Thanks to the boundary conditions, we know that $b_0=\alpha_0$ and $b_M=\beta_0$. Moreover, we set

$$v_i(x)=\sum_{l=4}^{N_i}a_{i,l}\psi_{i,l}(x), \qquad \text{for} \quad 1\le i\le M.$$

Accordingly, we expand the numerical solution $u_{\mathbf{N}}(x)$ as

$$\begin{aligned}u_{\mathbf{N}}(x)=&\sum_{i=1}^{M}v_i(x)+\sum_{i=1}^{M-1}b_i\sigma_i(x)+\sum_{i=1}^{M-1}c_i\lambda_i(x)+c_0\xi_1(x)\\ &+c_M\xi_M(x)+\alpha_0\zeta_1(x)+\beta_0\zeta_M(x).\end{aligned} \tag{5.8}$$

Let

$$A(v,\phi)=(\partial_x^2 v,\partial_x^2\phi)+(v,\phi), \qquad g(\phi)=(f,\phi)+\beta_2\partial_x\phi(b)-\alpha_2\partial_x\phi(a).$$

Then (3.3) can be rewritten as

$$A(u_{\mathbf{N}},\phi)=g(\phi), \qquad \forall\phi\in\bar{V}_{\mathbf{N}}(I). \tag{5.9}$$

Putting $\phi=\psi_{i,l'}$, $4\le l'\le N_i$, $1\le i\le M$, in (5.9), we obtain that

$$\begin{aligned}A(v_i,\psi_{i,l'})=&g(\psi_{i,l'})-b_{i-1}A(\sigma_{i-1},\psi_{i,l'})-b_iA(\sigma_i,\psi_{i,l'})-c_{i-1}A(\lambda_{i-1},\psi_{i,l'})\\ &-c_iA(\lambda_i,\psi_{i,l'}), \qquad 4\le l'\le N_i, \quad 2\le i\le M-1,\end{aligned} \tag{5.10}$$

$$\begin{aligned}A(v_1,\psi_{1,l'})=&g(\psi_{1,l'})-b_1A(\sigma_1,\psi_{1,l'})-c_1A(\lambda_1,\psi_{1,l'})-c_0A(\xi_1,\psi_{1,l'})\\ &-\alpha_0A(\zeta_1,\psi_{1,l'}), \qquad 4\le l'\le N_1,\end{aligned} \tag{5.11}$$

$$\begin{aligned}A(v_M,\psi_{M,l'})=&g(\psi_{M,l'})-b_{M-1}A(\sigma_{M-1},\psi_{M,l'})-c_{M-1}A(\lambda_{M-1},\psi_{M,l'})\\ &-c_MA(\xi_M,\psi_{M,l'})-\beta_0A(\zeta_M,\psi_{M,l'}), \qquad 4\le l'\le N_M.\end{aligned} \tag{5.12}$$

Besides, we take $\phi = \sigma_{i'}, \lambda_{i'},\ 1 \le i' \le M-1$, and $\phi = \xi_1, \xi_M$ in (5.9). Then

$$A(u_{\mathbf{N}}, \sigma_{i'}) = g(\sigma_{i'}), \qquad A(u_{\mathbf{N}}, \lambda_{i'}) = g(\lambda_{i'}), \tag{5.13}$$

$$A(u_{\mathbf{N}}, \xi_1) = g(\xi_1), \qquad A(u_{\mathbf{N}}, \xi_M) = g(\xi_M). \tag{5.14}$$

We are going to resolve the system (5.10)-(5.14). We shall provide an algorithm, which is also very efficient for related unsteady problems. To do this, we introduce the auxiliary functions

$$v_i^{(k)}(x) = \sum_{l=4}^{N_i} a_{i,l}^{(k)} \psi_{i,l}(x), \qquad \text{for} \quad 0 \le k \le 4, \quad 2 \le i \le M-1,$$

with

$$A(v_i^{(0)}, \psi_{i,l'}) = g(\psi_{i,l'}), \qquad A(v_i^{(1)}, \psi_{i,l'}) = -A(\sigma_{i-1}, \psi_{i,l'}), \tag{5.15}$$

$$A(v_i^{(2)}, \psi_{i,l'}) = -A(\sigma_i, \psi_{i,l'}), \quad A(v_i^{(3)}, \psi_{i,l'}) = -A(\lambda_{i-1}, \psi_{i,l'}), \tag{5.16}$$

$$A(v_i^{(4)}, \psi_{i,l'}) = -A(\lambda_i, \psi_{i,l'}). \tag{5.17}$$

Then, with the aid of (5.10), we have

$$v_i(x) = v_i^{(0)}(x) + b_{i-1}v_i^{(1)}(x) + b_i v_i^{(2)}(x) + c_{i-1}v_i^{(3)}(x) + c_i v_i^{(4)}(x), \quad 2 \le i \le M-1. \tag{5.18}$$

Next, we introduce the auxiliary functions

$$v_1^{(k)}(x) = \sum_{l=4}^{N_1} a_{1,l}^{(k)} \psi_{1,l}(x), \qquad k = 0, 2, 4, 5, 7,$$

with

$$A(v_1^{(0)}, \psi_{1,l'}) = g(\psi_{1,l'}), \qquad A(v_1^{(2)}, \psi_{1,l'}) = -A(\sigma_1, \psi_{1,l'}), \tag{5.19}$$

$$A(v_1^{(4)}, \psi_{1,l'}) = -A(\lambda_1, \psi_{1,l'}), \qquad A(v_1^{(5)}, \psi_{1,l'}) = -A(\xi_1, \psi_{1,l'}), \tag{5.20}$$

$$A(v_1^{(7)}, \psi_{1,l'}) = -A(\zeta_1, \psi_{1,l'}). \tag{5.21}$$

Then by virtue of (5.11),

$$v_1(x) = v_1^{(0)}(x) + b_1 v_1^{(2)}(x) + c_1 v_1^{(4)}(x) + c_0 v_1^{(5)}(x) + \alpha_0 v_1^{(7)}(x). \tag{5.22}$$

Similarly, we introduce the auxiliary functions

$$v_M^{(k)}(x) = \sum_{l=4}^{N_M} a_{M,l}^{(k)} \psi_{M,l}(x), \qquad k = 0, 1, 3, 6, 8,$$

with

$$A(v_M^{(0)}, \psi_{M,l'}) = g(\psi_{M,l'}), \qquad A(v_M^{(1)}, \psi_{M,l'}) = -A(\sigma_{M-1}, \psi_{M,l'}), \tag{5.23}$$

$$A(v_M^{(3)}, \psi_{M,l'}) = -A(\lambda_{M-1}, \psi_{M,l'}), \quad A(v_M^{(6)}, \psi_{M,l'}) = -A(\xi_M, \psi_{M,l'}), \tag{5.24}$$

$$A(v_M^{(8)}, \psi_{M,l'}) = -A(\zeta_M, \psi_{M,l'}). \tag{5.25}$$

Consequently, we obtain from (5.12) that

$$v_M(x) = v_M^{(0)}(x) + b_{M-1}v_M^{(1)}(x) + c_{M-1}v_M^{(3)}(x) + c_M v_M^{(6)}(x) + \beta_0 v_M^{(8)}(x). \tag{5.26}$$

Furthermore, let

$$G(x)=\sum_{i=1}^{M}v_i^{(0)}(x)+\alpha_0v_1^{(7)}(x)+\beta_0v_M^{(8)}(x)+\alpha_0\zeta_1(x)+\beta_0\zeta_M(x). \tag{5.27}$$

By substituting (5.18), (5.22), (5.26) and (5.27) into (5.8), we obtain

$$\begin{aligned}u_{\mathbf{N}}(x)=&\sum_{i=1}^{M-1}\left(b_i\Big(v_{i+1}^{(1)}(x)+v_i^{(2)}(x)+\sigma_i(x)\Big)+c_i\Big(v_{i+1}^{(3)}(x)+v_i^{(4)}(x)+\lambda_i(x)\Big)\right)\\&+c_0\Big(v_1^{(5)}(x)+\xi_1(x)\Big)+c_M\Big(v_M^{(6)}(x)+\xi_M(x)\Big)+G(x).\end{aligned} \tag{5.28}$$

In actual computation, we first solve the equations (5.15)-(5.17), (5.19)-(5.21) and (5.23)-(5.25), and then evaluate $G(x)$ explicitly. Next, by instituting (5.28) into (5.13) and (5.14), we derive a system with only $2M$ unknown values $b_i$ $(1\le i\le M-1)$ and $c_i$ $(0\le i\le M)$. This is much easier to be resolved. Finally, we obtain directly the numerical solution $u_{\mathbf{N}}(x)$ by using (5.28).

**Remark 5.1.** We can resolve the three systems (5.15)-(5.17), (5.19)-(5.21) and (5.23)-(5.25) separately. Since the basis function $\psi_{i,l}(x)$ vanishes outside $I_i$, it is very simple to be carried out for spectral element scheme (3.3). Moreover, the above three systems do not depend on the data $f,\alpha_0,\beta_0,\alpha_2$ and $\beta_2$. Thus, we could calculate $v_i^{(k)}(1\le k\le 4,\ 2\le i\le M-1)$, $v_1^{(k)}(k=2,4,5,7,9)$ and $v_M^{(k)}(k=1,3,6,8,10)$, in advance. This fact also simplifies computation essentially and saves much work. In particular, it saves a lot of computational time for the corresponding evolutionary problems. Obviously, it is very suitable for parallel computation too.

**Remark 5.2.** For one-dimensional problems, we may reform original problems to homogeneous problems. In this case, we do not need to use the basis functions $\zeta_1(x)$ and $\zeta_M(x)$. However, for multiple-dimensional problems, it is not easy to find proper variable transformation.

We now solve problem (3.2) with $a=-1$ and $b=1$, by using scheme (3.3) with the uniform mode $N_i=N$ and the uniform step size $h_i=h,\ 1\le i\le M$. The test function

$$U(x)=(10x^5+5)\sin kx,\qquad k\ne 0. \tag{5.29}$$

We measure the numerical errors by the maximum norm as follows,

$$E_{\mathbf{N}}=\max_{1\le i\le M,\ 1\le j\le N}\left|U\left(\frac{1}{2}(x_{i-1}+x_i)+\frac{h_i}{2}\delta_j^N\right)-u_{\mathbf{N}}\left(\frac{1}{2}(x_{i-1}+x_i)+\frac{h_i}{2}\delta_j^N\right)\right|.$$

We first take $k=1$ in (5.29), and list the errors $E_{\mathbf{N}}$ in Table 5.1. Clearly, the numerical errors decay very fast as $M=\frac{2}{h}$ and $N$ increase. This conforms the analysis very well.

We next take $k=10$ in (5.29). Since the test solution oscillates seriously, the standard spectral method ($M=1$) and the usual finite element method (small $N$) could not provide accurate numerical results. But, the spectral element method (3.3) might give very good numerical results. In Table 5.2, we list the numerical results with different $M$ and $N$. The numerical errors decay very fast as $M$ and $N$ increase, even the test function (5.29) oscillates seriously. This conforms the analysis again.

Table 5.1: Numerical errors of scheme (3.3) with $k = 1$.

| | $N = 6$ | $N = 10$ | $N = 14$ |
|---|---|---|---|
| $M = 1$ | 4.37E-02 | 3.14E-06 | 2.40E-11 |
| $M = 2$ | 2.01E-03 | 1.34E-08 | 6.44E-14 |
| $M = 4$ | 1.77E-05 | 8.39E-12 | 2.07E-14 |

Table 5.2: Numerical errors of scheme (3.3) with $k = 10$.

| | $N = 6$ | $N = 10$ | $N = 14$ | $N = 18$ |
|---|---|---|---|---|
| $M = 1$ | 3.13E+02 | 7.26E+00 | 1.11E-01 | 4.43E-03 |
| $M = 2$ | 1.51E+00 | 1.96E-02 | 9.76E-05 | 8.14E-08 |
| $M = 4$ | 2.76E-02 | 4.49E-05 | 7.48E-09 | 2.29E-12 |
| $M = 8$ | 9.97E-04 | 7.60E-09 | 5.30E-13 | 1.80E-13 |

We now turn to the spectral scheme (3.13). In this case, we need six kinds of base functions. The local polynomials corresponding to the subintervals $I_i(1 \leq i \leq M)$ are as follows,

$$\widetilde{\psi}_{i,l}(x) = \begin{cases} Y_{i,l}^{(m_i,n_i)}(x), & x \in \bar{I}_i, \\ 0, & \text{otherwise}. \end{cases} \tag{5.30}$$

The choice of parameters $m_i$ and $n_i$ depends on $M$. For $M = 1$, we take $m_1 = n_1 = 3$. For $M = 2$, we take $m_1 = n_2 = 2$ and $n_1 = m_2 = 3$. If $M \geq 3$, then we take $m_1 = n_M = 2, n_1 = m_M = 3$, and $m_i = n_i = 2$ for $2 \leq i \leq M-1$. In addition, $5 \leq l \leq N_i$ for $i = 1, M$, and $4 \leq l \leq N_i$ for $2 \leq i \leq M-1$. The sets of $\widetilde{\psi}_{i,l}(x)$ form the basis of trail function spaces $Q_{N_1}^{(2,3)}(I_1), Q_{N_M}^{(3,2)}(I_M)$ and $Q_{N_i}^{(2,2)}(I_i)$ for $2 \leq i \leq M-1$, respectively.

The basis functions for matching approximated functions at the interior nodes $x_i(1 \leq i \leq M-1)$ are

$$\widetilde{\sigma}_i(x) = \begin{cases} q_{m_i,n_i,0}^{i,+}(x), & x \in \bar{I}_i, \\ q_{m_{i+1},n_{i+1},0}^{i+1,-}(x), & x \in \bar{I}_{i+1}, \\ 0, & \text{otherwise}. \end{cases} \tag{5.31}$$

The choice of parameters $m_i$ and $n_i$ depends on $M$. For $M = 2$, we take $m_1 = n_2 = 2$ and $n_1 = m_2 = 3$. If $M \geq 3$, then we take $m_1 = n_M = 2$, $n_1 = m_M = 3$, and $m_i = n_i = 2$ for $2 \leq i \leq M-1$. Obviously, $\widetilde{\sigma}_i(x_i) = 1$ and $\partial_x \widetilde{\sigma}_i(x_i) = 0$, $1 \leq i \leq M-1$.

The basis functions for matching the first order derivatives of underlying functions at the interior nodes $x_i(1 \leq i \leq M-1)$ are

$$\widetilde{\lambda}_i(x) = \begin{cases} \frac{h_i}{2} q_{m_i,n_i,1}^{i,+}(x), & x \in \bar{I}_i, \\ \frac{h_{i+1}}{2} q_{m_{i+1},n_{i+1},1}^{i+1,-}(x), & x \in \bar{I}_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{5.32}$$

where $m_i$ and $n_i$ are the same as for $\widetilde{\sigma}_i(x)$. Evidently, $\widetilde{\lambda}_i(x_i) = 0$ and $\partial_x \widetilde{\lambda}_i(x_i) = 1$, $1 \leq i \leq M-1$.

The basis functions for matching approximated functions at the boundary ($x = x_0, x_M$) are

$$\widetilde{\zeta}_1(x) = \begin{cases} q_{m_1,n_1,0}^{1,-}(x), & x \in \bar{I}_1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.33}$$

$$\widetilde{\zeta}_M(x) = \begin{cases} q^{M,+}_{m_M,n_M,0}(x), & x \in \bar{I}_M, \\ 0, & \text{otherwise}, \end{cases} \tag{5.34}$$

where $m_1 = n_1 = m_M = n_M = 3$, for $M = 1$, while $m_1 = n_M = 2$ and $n_1 = m_M = 3$, for $M \geq 2$.

The basis functions for matching the first order derivatives of underlying functions at the boundary $(x = x_0, x_M)$ are

$$\widetilde{\xi}_1(x) = \begin{cases} \frac{h_1}{2} q^{1,-}_{m_1,n_1,1}(x), & x \in \bar{I}_1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.35}$$

$$\widetilde{\xi}_M(x) = \begin{cases} \frac{h_M}{2} q^{M,+}_{m_M,n_M,1}(x), & x \in \bar{I}_M, \\ 0, & \text{otherwise}, \end{cases} \tag{5.36}$$

where $m_1, n_1, m_M$ and $n_M$ are the same as in (5.33) and (5.34).

In addition, we take the basis functions for matching the second order derivatives of underlying functions at the boundary $(x = x_0, x_M)$, as

$$\widetilde{\eta}_1(x) = \begin{cases} \frac{h_1^2}{4} q^{1,-}_{m_1,n_1,2}(x), & x \in \bar{I}_1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.37}$$

$$\widetilde{\eta}_M(x) = \begin{cases} \frac{h_M^2}{4} q^{M,+}_{m_M,n_M,2}(x), & x \in \bar{I}_M, \\ 0, & \text{otherwise}, \end{cases} \tag{5.38}$$

where $m_1, n_1, m_M$ and $n_M$ are the same as in (5.33) and (5.34).

Furthermore, we let

$$\begin{aligned} \widetilde{v}_i(x) &= \sum_{l=5}^{N_i} a_{i,l}\widetilde{\psi}_{i,l}(x), && \text{for} \quad i = 1, M, \\ \widetilde{v}_i(x) &= \sum_{l=4}^{N_i} a_{i,l}\widetilde{\psi}_{i,l}(x), && \text{for} \quad 2 \leq i \leq M-1. \end{aligned}$$

We expand the numerical solution $u_{\mathbf{N}}(x)$ as

$$\begin{aligned} u_{\mathbf{N}}(x) = \sum_{i=1}^{M} \widetilde{v}_i(x) + \sum_{i=1}^{M-1} b_i\widetilde{\sigma}_i(x) + \sum_{i=1}^{M-1} c_i\widetilde{\lambda}_i(x) + c_0\widetilde{\xi}_1(x) + c_M\widetilde{\xi}_M(x) \\ + \alpha_0\widetilde{\zeta}_1(x) + \beta_0\widetilde{\zeta}_M(x) + \alpha_2\widetilde{\eta}_1(x) + \beta_2\widetilde{\eta}_M(x), \end{aligned} \tag{5.39}$$

where $b_i, c_i$ are the same as before.

Let $A(v, \phi)$ and $g(\phi)$ be same as in (5.9). Then the scheme (3.13) can be rewritten as

$$A(u_{\mathbf{N}}, \phi) = g(\phi), \qquad \forall \phi \in \bar{\bar{V}}_{\mathbf{N}}(I). \tag{5.40}$$

Table 5.3: Numerical errors of scheme (3.13) with $k = 1$.

| | $N = 6$ | $N = 10$ | $N = 14$ |
|---|---|---|---|
| $M = 1$ | 6.61E-02 | 3.97E-06 | 2.82E-11 |
| $M = 2$ | 3.07E-03 | 1.65E-08 | 5.13E-14 |
| $M = 4$ | 2.68E-05 | 1.03E-11 | 7.99E-15 |

Table 5.4: Numerical errors of scheme (3.13) with $k = 10$.

| | $N = 6$ | $N = 10$ | $N = 14$ | $N = 18$ |
|---|---|---|---|---|
| $M = 1$ | 2.99E+02 | 7.26E+00 | 1.28E-01 | 4.90E-03 |
| $M = 2$ | 2.55E+00 | 2.34E-02 | 1.16E-04 | 9.21E-08 |
| $M = 4$ | 2.66E-02 | 5.45E-05 | 8.62E-09 | 2.03E-12 |
| $M = 8$ | 1.53E-03 | 4.61E-09 | 3.72E-13 | 5.02E-13 |

We can resolve (5.40) in the same way as for (5.9).

In Tables 5.3 and 5.4, we list the numerical errors of scheme (3.10) for solving problem (3.2) with the test function (5.29), $k = 1, 10$. We find that the spectral element method with essential imposition of boundary conditions works well even for oscillated solutions.

It is noted that some authors provided certain other base functions and interpolations for matching Nuemann and Robin boundary conditions exactly, see [28, 29].

## 6. Concluding Remarks

In this paper, we proposed the new spectral and spectral element methods for high order problems with mixed inhomogeneous boundary conditions. We derived the error estimates precisely. We also provided the spectral element method with essential imposition of mixed boundary conditions. The numerical results indicated the high accuracy of suggested algorithm, which works well even for oscillating solutions. Although we only considered two model problems in this work, the idea and techniques developed in this work are also useful for a large class of spectral and spectral element methods for other high order problems with various boundary conditions.

The main idea and technology developed in this paper could be extended to multi-dimensional problems. Recently, Yu and Guo [31] considered the Legendre quasi-orthogonal approximation in two-dimensional space with its application to spectral element method for fourth order problems. For example, let domain $\Omega$ be a union of several rectangles, with the boundary $\partial\Omega = \partial^*\Omega \cup \partial^{**}\Omega, \partial^*\Omega \cap \partial^{**}\Omega = \emptyset$. Let $d$ and $\beta$ be nonnegative constants. Yu and Guo [31] proposed the spectral element method for following model problem,

$$\begin{cases} \Delta^2 U(\mathbf{x}) + dU(\mathbf{x}) = F(\mathbf{x}), & \text{in } \Omega, \\ \Delta U(\mathbf{x}) + \beta\partial_n U(\mathbf{x}) = G_2(\mathbf{x}), & \text{on } \partial^{**}\Omega, \\ \partial_n U(\mathbf{x}) = G_1(\mathbf{x}), & \text{on } \partial^*\Omega, \\ U(\mathbf{x}) = G_0(\mathbf{x}), & \text{on } \partial\Omega. \end{cases}$$

**Acknowledgments.** The work of the first author was supported by NSF of China N.11171227, Fund for Doctor Degree Authority of Chinese Educational Ministry N.20123127110001, Fund for E-institute of Shanghai Universities N.E03004, and Leading Academic Discipline Project of Shanghai Municipal Education Commission N.J50101. The work of the second author was supported by NSF of China N.11171227 and N.11326244, and Fund for Young Teachers of Shanghai Universities N.ZZshjr12009. The work of the third author was supported by NSF of China N.11171227 and N.11371123, Research Fund for Young Teachers of Jiangsu Normal University N.11XLR27, and Priority Academic Program Development of Jiangsu Higher Education Institutions.

## References


[1] F. Auteri, N. Parolini and L. Quartapelle, Essential imposition of Neumann condition in Galerkin-Legendre elliptic solver, *J. Comput. Phys.,* **185** (2003), 427-444.

[2] Z. Belhachmi, C. Bernardi and A. Karageorghis, Spectral element discretization of the circular driven cavity, Part II: the bilaplacian equation. *SIAM J. Numer. Anal.*, **38** (2001), 1926-1960.

[3] C. Bernardi and Y. Maday, Spectral Methods, in Handbook of Numerical Analysis, Vol.5, Techniques of Scientific Computing, 209-486, edited by P. G. Ciarlet and J. L. Lions, Elsevier, Amsterdam, 1997.

[4] B. Bialecki and A. Karageorghis, A Legendre spectral Galerkin method for the biharmonic Dirichlet problem, *SIAM J. Sci. Comput.,* **22** (2000), 1549-1569.

[5] P. E. Bjørstad and B. P. Tjøstheim, Efficient algorithms for solving a fourth-order equation with spectral-Galerkin method, *SIAM J. Sci. Comput.,* **18** (1997), 621-632.

[6] C. Canuto, M.Y. Hussaini, A. Quarteroni and T. A. Zang, Spectral Methods, Fundamentals in Single Domains, Springer, Berlin, 2006.

[7] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, Spectral Methods: Evolution to complex Geometries and Applications to Fluid Dynamics, Springer-Verlag, Berlin, 2007.

[8] E.H. Doha and A.H. Bhrawy, Efficient spectral-Galerkin algorithms for direct solution of fourth-order differential equations using Jacobi polynomials, *Appl. Numer. Math.*, **58** (2008), 1224-1244.

[9] E.H. Doha and A. H. Bhrawy, An efficient direct solver for multidimensional elliptic Robin boundary value problems using a Legendre spectral-Galerkin method, *Comput. and Math. with Appl.,* **64** (2012), 558-571.

[10] A. Ezzirani and A. Guessab, A fast algorithm for Gaussian type quadrature formulae with mixed boundary conditions and some lumped mass spectral approximations, *Math. Comput.*, **225** (1999), 217-248.

[11] D. Gottlieb and S.A. Orszag, Numerical Analysis of Spectral Methods: Theory and Applications, SIAM-CBMS, Philadelphia, 1977.

[12] Ben-yu Guo, Spectral Methods and Their Applications, World Scientific, Singapore, 1998.

[13] Ben-yu Guo, Gegenbauer approximation in certain Hilbert spaces and its applications to singular differential equations on the whole line, *SIAM J. Numer. Anal.*, **37** (2000), 621-645.

[14] Ben-yu Guo, Jacobi approximations in certain Hilbert spaces and their applications to singular differential equations, *J. Math. Anal. and Appl.*, **243** (2000), 373-408.

[15] Ben-yu Guo and Li-ping He, The fully discrete Legendre spectral approximation of two-dimensional unsteady incompressible fluid flow in stream function form, *SIAM J. Numer. Anal.,* **35** (1998), 146-176.

[16] Ben-yu Guo and Hong-li Jia, Spectral method on quadrilaterals, *Math. Comput.,* **79** (2010), 2237-2264.

[17] Ben-yu Guo, Jie Shen and Li-lian Wang, Optimal spectral-Galerkin methods using generalized Jacobi polynomials, *J. Sci. Comput.*, **27** (2006), 305-322.

[18] Ben-yu Guo, Jie Shen and Li-lian Wang, Generalized Jacobi polynomials/Functions and their applications, *Appl. Numer. Math.,* **59** (2009), 1011-1028.

[19] Ben-yu Guo, Tao Sun and Chao Zhang, Jacobi and Laguerre quasi-orthogonal approximations and related interpolations, *Math. Comput.,* **82** (2013), 413-441.

[20] Ben-yu Guo, and Li-lian Wang, Jacobi approximations and Jacobi-Gauss-type interpolations in non-uniformly Jacobi-weighted Sobolev spaces, *J. Appr. Theo.*, **28** (2004), 1-41.

[21] Ben-yu Guo and Tian-jun Wang, Composite generalized Laguerre-Legendre spectral method with domain decomposition and its application to Fokker-Planck equation in an finite channel, *Math. Comput.*, **78** (2009), 129-151.

[22] Ben-yu Guo, Zhong-qing Wang, Zheng-su Wan and Delin Chu, Second order Jacobi approximation with applications to fourth order differential equations, *Appl. Numer. Math.,* **55** (2005), 480-502.

[23] Ben-yu Guo and Ke-ji Zhang, On non-isotropic Jacobi pseudospectral method, *J. of Comput. Math.*, **26** (2008), 511-535.

[24] G. Karniadakis and S. J. Sherwin, Spectral/hp Element Methods for CFD, Oxford University Press, Oxford, 1999.

[25] He-ping Ma and Wei-wei Sun, A Legendre-Petrov-Galerkin and Chebyshev collocation method for third-order differential equations, *SIAM J. Numer. Anal.,* **38** (2000), 1425-1438.

[26] He-ping Ma and Wei-wei Sun, Optimal error estimates of the Legendre-Petrov-Galerkin method for the Korteweg-de Vries equation, *SIAM J. Numer. Anal.,* **39** (2001), 1380-1394.

[27] Jie Shen, Efficient spectral-Galerkin method, I. Direct solvers of second- and fourth-order equations using Legendre polynomials, *SIAM J. Sci. Comput.,* **15** (1994), 1489-1505.

[28] Jie Shen and Tao Tang, Spectral and High-order Methods with Applications, Scinence Press, Beijing, 2006.

[29] Li-lian Wang and Ben-yu Guo, Interpolation approximations based on Gauss-Lobatto-Legendre-Birkhoff quadrature, *J. Appro. Theor.,* **161** (2009), 142-173.

[30] Zhong-qing Wang and Li-lian Wang, A collocation method with exact imposition of mixed boundary conditions, *J. Sci. Comput.*, **42** (2010), 291-317.

[31] Xu-hong Yu and Ben-yu Guo, Spectral element method for mixed inhomogeneous boundary value problems of fourth order, *J. Sci. Comput.*, DOI: 10.1007/s10915-014-9844-y.