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POISSON PRECONDITIONING FOR SELF-ADJOINT ELLIPTIC PROBLEMS*

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Abstract

In this paper, we formulate interface problem and Neumann elliptic boundary value problem into a form of linear operator equations with self-adjoint positive definite operators. We prove that in the discrete level the condition number of these operators is independent of the mesh size. Therefore, given a prescribed error tolerance, the classical conjugate gradient algorithm converges within a fixed number of iterations. The main computation task at each iteration is to solve a Dirichlet Poisson boundary value problem in a rectangular domain, which can be furnished with fast Poisson solver. The overall computational complexity is essentially of linear scaling.

Mathematics subject classification: 65N30, 65T50.
Key words: Fast Poisson solver, Interface problem, Self-adjoint elliptic problem, Conjugate gradient method.

1. Introduction

Self-adjoint elliptic problem can be reformulated as some Riesz representation in an appropriate Hilbert space. To clarify this point, let us consider the Dirichlet boundary value problem

$$\begin{aligned} &-\nabla\cdot(\beta(x)\nabla u)+c(x)u=f, \quad \forall\, x\in\Omega, \\ &u=0, \quad \forall\, x\in\partial\Omega. \end{aligned}$$

where Ω is a bounded domain of dimension d, and $\beta(x)$, $c(x)$ and f are given functions in Ω. The associated variational problem is to find a distribution $u\in H_0^1(\Omega)$ such that

$$a_{var,\Omega}(u,v)\overset{def}{=}(\beta(x)\nabla u,\nabla v)_\Omega+(c(x)u,v)_\Omega=(f,v)_\Omega, \quad \forall\, v\in H_0^1(\Omega). \tag{1.1}$$

Here $(\cdot,\cdot)_\Omega$ denotes the standard L^2-inner product in the domain Ω. If the coefficient functions $\beta(x)$ and $c(x)$ satisfy

$$0<\beta_0\le\beta(x)\le\beta_1<\infty, \quad 0\le c(x)\le c_{max}<\infty, \quad \forall\, x\in\Omega, \tag{1.2}$$

where β_0, β_1 and c_{max} are three constants, then the bilinear form $a_{var,\Omega}(\cdot,\cdot)$ defines an inner product in $H_0^1(\Omega)$. The weak solution u is simply the Riesz representation of functional $(f,v)_\Omega$ with respect to the inner product $a_{var,\Omega}(\cdot,\cdot)$ in $H_0^1(\Omega)$.

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There might be different inner products in a same vector space. In some cases, it is possible to choose an equivalent reference inner product, such that the Riesz representation with respect to this reference inner product is simpler. For example, the bilinear form

$$a_{0,\Omega}(u,v) = (\nabla u, \nabla v)_\Omega$$

gives an equivalent inner product as $a_{var,\Omega}(\cdot,\cdot)$ in $H_0^1(\Omega)$, but the Riesz representation with respect to $a_{0,\Omega}(\cdot,\cdot)$ is simpler since this corresponds to a Dirichlet Poisson problem. By introducing the representation operator T as

$$a_{var,\Omega}(u,v) = a_{0,\Omega}(Tu,v), \quad \forall\, u,v \in H_0^1(\Omega),$$

the variational problem (1.1) can be rewritten into a form of operator equation

$$Tu = R_f. \tag{1.3}$$

In the above, R_f denotes the Riesz representation of functional $(f,v)_\Omega$ with respect to the reference inner product $a_{0,\Omega}(\cdot,\cdot)$. Since $a_{0,\Omega}(\cdot,\cdot)$ is equivalent to $a_{var,\Omega}(\cdot,\cdot)$, T is both self-adjoint and positive definite. Obviously, these properties are inherited automatically in the discrete level, and the bounds of operator T are independent of the mesh size when a conforming finite element method is used. This implies that the operator Eq. (1.3), thus the original problem (1.1), can be solved by the *Conjugate Gradient* (CG) method within a fixed number of iterations. At each iteration, one needs to determine a Riesz representation of some functional with respect to the reference inner product. If this can be achieved with an essentially linear scaling algorithm, such as the fast Poisson solver for the model problem when the domain is rectangular, the overall scheme based on CG iterations is then essentially of linear scaling.

There are two ingredients involved in the above solution strategy. The first one is how to formulate a self-adjoint elliptic problem into a Riesz representation problem. The second one is how to determine an equivalent reference inner product such that the Riesz representation can be derived with a linear scaling algorithm. Needless to say, these issues are coupled together and problem dependent. We need to study them case by case.

Interface problem is ubiquitous in fluid dynamics and material science. It has been a hot research subject for many years. The main difficulty for solving interface problem is due to the fact that the solution is generally not smooth globally, thus the traditional finite difference method (FDM) works poorly near the interface. As early as in 1977, Peskin [9] proposed the immersed boundary (IB) method to handle the singular interface force in his blood flow model for heart. His basic idea is to approximate the singular delta function with a smoother delta series. In this way, the singular force is smeared out, and the standard FDM is then applicable. The IB method has been extended in a great deal, and become very popular in the simulation of interface-related problems. The readers are referred to [10] for more detailed information.

Despite the overwhelming success, the IB method is criticized due to the less satisfying accuracy. This motivated Leveque and Li [5,6] to develop the immersed interface method (IIM). The original version of IIM is formally second order accurate but results in a linear system with non-symmetric coefficient matrix. This unpleasant fact has a subtle influence on the convergence of their proposed iterative scheme [4]. Later, Li and Ito [7] proposed some maximum principle preserving schemes to avoid this convergence problem. In more recent years, the finite element version of IIM [2,3,13] has been studied more extensively. In comparison to the FDM, the finite element method (FEM) has two remarkable features. First, the FEM can handle complicated