Journal of Computational Mathematics
Vol.32, No.5, 2014, 560–578.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1405-m4293

POISSON PRECONDITIONING FOR SELF-ADJOINT ELLIPTIC PROBLEMS*

Houde Han Chunxiong Zheng
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
Email: hhan@math.tsinghua.edu.cn czheng@math.tsinghua.edu.cn

Abstract

In this paper, we formulate interface problem and Neumann elliptic boundary value problem into a form of linear operator equations with self-adjoint positive definite operators. We prove that in the discrete level the condition number of these operators is independent of the mesh size. Therefore, given a prescribed error tolerance, the classical conjugate gradient algorithm converges within a fixed number of iterations. The main computation task at each iteration is to solve a Dirichlet Poisson boundary value problem in a rectangular domain, which can be furnished with fast Poisson solver. The overall computational complexity is essentially of linear scaling.

Mathematics subject classification: 65N30, 65T50.
Key words: Fast Poisson solver, Interface problem, Self-adjoint elliptic problem, Conjugate gradient method.

1. Introduction

Self-adjoint elliptic problem can be reformulated as some Riesz representation in an appropriate Hilbert space. To clarify this point, let us consider the Dirichlet boundary value problem

$$\begin{aligned} &-\nabla\cdot(\beta(x)\nabla u)+c(x)u=f, \quad \forall\, x\in\Omega,\\ &u=0, \quad \forall\, x\in\partial\Omega. \end{aligned}$$

where Ω is a bounded domain of dimension d, and $\beta(x)$, $c(x)$ and f are given functions in Ω. The associated variational problem is to find a distribution $u\in H_0^1(\Omega)$ such that

$$a_{var,\Omega}(u,v)\stackrel{def}{=}(\beta(x)\nabla u,\nabla v)_\Omega+(c(x)u,v)_\Omega=(f,v)_\Omega, \quad \forall\, v\in H_0^1(\Omega). \tag{1.1}$$

Here $(\cdot,\cdot)_\Omega$ denotes the standard L^2-inner product in the domain Ω. If the coefficient functions $\beta(x)$ and $c(x)$ satisfy

$$0<\beta_0\le\beta(x)\le\beta_1<\infty, \quad 0\le c(x)\le c_{max}<\infty, \quad \forall\, x\in\Omega, \tag{1.2}$$

where β_0, β_1 and c_{max} are three constants, then the bilinear form $a_{var,\Omega}(\cdot,\cdot)$ defines an inner product in $H_0^1(\Omega)$. The weak solution u is simply the Riesz representation of functional $(f,v)_\Omega$ with respect to the inner product $a_{var,\Omega}(\cdot,\cdot)$ in $H_0^1(\Omega)$.

* Received October 14, 2012 / Revised version received March 28, 2014 / Accepted May 26, 2014 / Published online August 22, 2014 /

There might be different inner products in a same vector space. In some cases, it is possible to choose an equivalent reference inner product, such that the Riesz representation with respect to this reference inner product is simpler. For example, the bilinear form

$$a_{0,\Omega}(u,v) = (\nabla u, \nabla v)_\Omega$$

gives an equivalent inner product as $a_{var,\Omega}(\cdot,\cdot)$ in $H_0^1(\Omega)$, but the Riesz representation with respect to $a_{0,\Omega}(\cdot,\cdot)$ is simpler since this corresponds to a Dirichlet Poisson problem. By introducing the representation operator T as

$$a_{var,\Omega}(u,v) = a_{0,\Omega}(Tu,v), \quad \forall\, u,v \in H_0^1(\Omega),$$

the variational problem (1.1) can be rewritten into a form of operator equation

$$Tu = R_f. \tag{1.3}$$

In the above, R_f denotes the Riesz representation of functional $(f,v)_\Omega$ with respect to the reference inner product $a_{0,\Omega}(\cdot,\cdot)$. Since $a_{0,\Omega}(\cdot,\cdot)$ is equivalent to $a_{var,\Omega}(\cdot,\cdot)$, T is both self-adjoint and positive definite. Obviously, these properties are inherited automatically in the discrete level, and the bounds of operator T are independent of the mesh size when a conforming finite element method is used. This implies that the operator Eq. (1.3), thus the original problem (1.1), can be solved by the *Conjugate Gradient* (CG) method within a fixed number of iterations. At each iteration, one needs to determine a Riesz representation of some functional with respect to the reference inner product. If this can be achieved with an essentially linear scaling algorithm, such as the fast Poisson solver for the model problem when the domain is rectangular, the overall scheme based on CG iterations is then essentially of linear scaling.

There are two ingredients involved in the above solution strategy. The first one is how to formulate a self-adjoint elliptic problem into a Riesz representation problem. The second one is how to determine an equivalent reference inner product such that the Riesz representation can be derived with a linear scaling algorithm. Needless to say, these issues are coupled together and problem dependent. We need to study them case by case.

Interface problem is ubiquitous in fluid dynamics and material science. It has been a hot research subject for many years. The main difficulty for solving interface problem is due to the fact that the solution is generally not smooth globally, thus the traditional finite difference method (FDM) works poorly near the interface. As early as in 1977, Peskin [9] proposed the immersed boundary (IB) method to handle the singular interface force in his blood flow model for heart. His basic idea is to approximate the singular delta function with a smoother delta series. In this way, the singular force is smeared out, and the standard FDM is then applicable. The IB method has been extended in a great deal, and become very popular in the simulation of interface-related problems. The readers are referred to [10] for more detailed information.

Despite the overwhelming success, the IB method is criticized due to the less satisfying accuracy. This motivated Leveque and Li [5,6] to develop the immersed interface method (IIM). The original version of IIM is formally second order accurate but results in a linear system with non-symmetric coefficient matrix. This unpleasant fact has a subtle influence on the convergence of their proposed iterative scheme [4]. Later, Li and Ito [7] proposed some maximum principle preserving schemes to avoid this convergence problem. In more recent years, the finite element version of IIM [2,3,13] has been studied more extensively. In comparison to the FDM, the finite element method (FEM) has two remarkable features. First, the FEM can handle complicated geometries without too many additional efforts. Second, the FEM maintain the variational structure of the original problem, so that the self-adjointness (if existing) is inherited naturally after discretization and the error analysis is generally easier.

In some cases, the solution of interface problem is at least piecewise smooth, though not globally. However, as pointed out and analyzed in [1], this is already sufficient to ensure a numerical solution with nearly optimal accuracy if an interface-fitted mesh is used. Besides, the coefficient matrix of the resulting linear system is maintained symmetric. Therefore, for those satisfied with the approximating accuracy and the symmetric feature of the resulting linear system, the interface problem is largely solved. In the authors' opinion, the key issue related to interface problem is the designing of fast algorithm based on a rigorous numerical analysis.

This was our original motivation to initiate this research. Inspired by the abstract solution strategy described at the beginning of this section, we develop a CG iteration algorithm based on the fast Poisson solver for the elliptic interface problem. In the language of preconditioning, we use constant coefficient Poisson equation to precondition the variable coefficient interface problem. We stick to the fast Poisson solver because it is essentially of linear scaling, and more importantly easy to be implemented. We prove that given a prescribed error tolerance, the CG method terminate after a fixed number of iterations. This ensures that the overall scheme is essentially of linear scaling. As an extension, the Neumann boundary value problem is also considered in this paper. Following the analogous idea, we formulate the boundary value problem into a variational problem such that the Poisson equation can be taken as a uniform preconditioner.

The rest of this paper is organized as follows. In Section 2, we recall the CG method and the fast Poisson algorithm with bilinear finite elements. In Section 3, we consider the interface problem. In Section 4, we consider the Neumann boundary value problems. Numerical tests are reported in Section 5, and Section 6 concludes this paper.

2. Conjugate Gradient Method and Fast Poisson Solver

We first recall some basic notions in functional analysis.

Definition 2.1. *Suppose V is a (real) Hilbert space with inner product $(\cdot,\cdot)$. The induced norm is denoted by $\|\cdot\|$. A linear operator $T : V \to V$ is regarded bounded if there exists a constant $c_2 > 0$ such that*

$$\|Tv\| \leq c_2\|v\|, \ \forall\, v \in V.$$

T is regarded coercive if there exists a constant $c_1 > 0$ such that

$$(Tv, v) \geq c_1\|v\|^2, \ \forall\, v \in V.$$

T is regarded self-adjoint if

$$(Tu, v) = (u, Tv), \ \forall\, u, v \in V.$$

If the linear operator T is bounded, coercive and self-adjoint, we call it Self-adjoint Positive Definite *(SPD). The constants c_1 and c_2 are regarded* bounds *of the SPD operator T.*

Obviously, the concept of SPD is an extension of that for a symmetric positive definite matrix. If T is SPD, then its spectrum $\sigma(T) \subset [c_1, c_2]$ where c_1 and c_2 are bounds of T. Define the condition number as

$$\kappa(T) = \frac{\sup \sigma(T)}{\inf \sigma(T)},$$

and it follows that $\kappa(T) \le c_2/c_1$.

Given a Hilbert space V with inner product $(\cdot,\cdot)_\alpha$, we consider the Riesz representation of a bounded linear functional $l(v)$. More precisely, we seek $u \in V$ such that

$$(u, v)_\alpha = l(v), \ \forall\, v \in V.$$

Suppose $(\cdot,\cdot)_{ref}$ is another equivalent inner product in V. Define the representation operator T by

$$(u, v)_\alpha = (Tu, v)_{ref}, \ \forall\, u, v \in V,$$

and let R_l be the Riesz representation of $l(v)$ under the reference inner product $(\cdot,\cdot)_{ref}$, i.e.,

$$l(v) = (R_l, v)_{ref}, \ \forall\, v \in V.$$

Then u solves the operator equation

$$Tu = R_l. \tag{2.1}$$

Since $(\cdot,\cdot)_\alpha$ and $(\cdot,\cdot)_{ref}$ are equivalent, T is SPD. The operator Eq. (2.1) can be solved with the standard CG method:

1. Given a relative residual error tolerance ϵ and initial guess $u^{(0)}$.
2. Set $r^{(0)} = R_l - Tu^{(0)}$, $p^{(0)} = r^{(0)}$.
3. for $k = 0, 1, \cdots$

$$\begin{aligned}
\alpha_k &= \frac{(r^{(k)}, r^{(k)})_{ref}}{(p^{(k)}, p^{(k)})_\alpha},\\
u^{(k+1)} &= u^{(k)} + \alpha_k p^{(k)},\\
r^{(k+1)} &= r^{(k)} - \alpha_k T p^{(k)},\\
\beta_k &= \frac{(r^{(k+1)}, r^{(k+1)})_{ref}}{(r^{(k)}, r^{(k)})_{ref}},\\
p^{(k+1)} &= r^{(k+1)} + \beta_k p^{(k)}.
\end{aligned}$$

4. Iterations terminate when $\|r^{(k)}\|_{ref}/\|r^{(0)}\|_{ref} \le \epsilon$.

The error bound at the k-th iteration is given by (see for example [11])

$$\frac{\|u - u^{(k)}\|_\alpha}{\|u - u^{(0)}\|_\alpha} \le 2\left(\frac{\sqrt{\kappa(T)} - 1}{\sqrt{\kappa(T)} + 1}\right)^k,$$

where $\kappa(T)$ is the condition number of T.

Fast Poisson solver is an efficient numerical algorithm to solve Dirichlet Poisson problems defined in a rectangular domain. There are several versions in the literature, but all of them share the same spirit. We recall this algorithm for the bilinear rectangular elements.

Given a rectangular domain $\Omega = \Omega_1 \times \cdots \times \Omega_d$ of dimension d, let us consider the discrete variational problem: Find $u \in V_h^R(\Omega)$ such that $u|_{\partial\Omega} = g \in \partial V_h^R(\Omega)$ and

$$a_{z,\Omega}(u, v) \overset{def}{=} (\nabla u, \nabla v)_\Omega + z(u, v)_\Omega = l(v), \quad \forall\, v \in V_{h,0}^R(\Omega). \tag{2.2}$$

Here and hereafter, $V_h^R(\Omega)$ denotes the bilinear *Rectangular* finite element space in Ω, $V_{h,0}^R(\Omega)$ the maximal subspace of $V_h^R(\Omega)$ with trace zero, $\partial V_h^R(\Omega)$ the trace space, l a bounded linear functional on $H^1(\Omega)$, and $(\cdot,\cdot)_\Omega$ the standard L^2 inner product. This variational problem arises when one discretizes the Poisson operator $-\Delta + zI$ equipped with Dirichlet boundary condition. If $z \in \mathbb{R}$ is not a Dirichlet eigenvalue of the discrete Laplace operator, the problem (2.2) is uniquely solvable.

Let us denote the hat basis functions in the i-th direction as $\psi_{i,k}(x_i)$, $k = 0, 1, \cdots, M_i$, $i = 1, \cdots, d$. Introducing the vectorial index $\mathbf{k} = (k_1, k_2, \cdots, k_d)$, we can label finite element basis functions as

$$\psi_{\mathbf{k}}(x) = \prod_{i=1,\cdots,d} \psi_{i,k_i}(x_i), \quad x = (x_1, \cdots, x_d)^\top.$$

By expanding u under this set of basis functions as

$$u = \sum_{\mathbf{k}} u_{\mathbf{k}} \psi_{\mathbf{k}},$$

the variational problem (2.2) can be reformulated as a linear system

$$\mathbf{Tu} = \mathbf{b}, \quad \mathbf{u} = (u_{\mathbf{k}}), \tag{2.3}$$

where $\mathbf{T} = (t_{\mathbf{k},\mathbf{k}'})$ is the reduced total stiffness matrix defined as

$$\begin{aligned} t_{\mathbf{k},\mathbf{k}'} &= a_{z,\Omega}(\psi_{\mathbf{k}}, \psi_{\mathbf{k}'}) \\ &= \sum_{i=1,\cdots,d} (\nabla\psi_{i,k_i}, \nabla\psi_{i,k_i'})_{\Omega_i} \prod_{j\neq i} (\psi_{j,k_j}, \psi_{j,k_j'})_{\Omega_j} + z \prod_{i=1,\cdots,d} (\psi_{i,k_i}, \psi_{i,k_i'})_{\Omega_i}, \end{aligned}$$

and $\mathbf{b} = (b_{\mathbf{k}})$ is the reduced load vector defined as

$$b_{\mathbf{k}} = l(\psi_{\mathbf{k}}) - \sum_{\mathbf{k}' \in \mathrm{BInd}} t_{\mathbf{k},\mathbf{k}'} u_{\mathbf{k}'}, \quad \forall\, \mathbf{k} \notin \mathrm{BInd}.$$

Here BInd denotes the set of boundary indices, and $u_{\mathbf{k}'}$ for $\mathbf{k}' \in \mathrm{BInd}$ is given by the Dirichlet boundary condition. It is easy to verify that the reduced stiffness and mass matrices in the i-th direction are of the form

$$T_i = \frac{1}{h_i} \begin{pmatrix} [r]2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdots & \cdots & \cdots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \quad U_i = h_i \begin{pmatrix} [r]\frac{2}{3} & \frac{1}{6} & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \\ & \cdots & \cdots & \cdots & \\ & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & & & \frac{1}{6} & \frac{2}{3} \end{pmatrix},$$

where h_i denotes the mesh size in the i-th direction. The key point is that these two matrices are simultaneously diagonalizable by the discrete sine transform

$$x = (x_1, \cdots, x_{M_i-1})^\top \to y = (y_1, \cdots, y_{M_i-1})^\top = V_i x$$

defined by

$$y_k = \sum_{j=1}^{M_i-1} \sin\left(\frac{kj\pi}{M_i}\right) x_j.$$

The diagonal elements of diagonal matrices $V_i T_i V_i^{-1}$ and $V_i U_i V_i^{-1}$ are, for $1 \leq k \leq M_i - 1$:

$$T_{i,k} = \frac{2 - 2\cos\left(\frac{k\pi}{M_i}\right)}{h_i}, \qquad U_{i,k} = h_i \left(\frac{2}{3} + \frac{\cos\left(\frac{k\pi}{M_i}\right)}{3}\right).$$

Since $\mathbf{T}$ is a summation of some tensor products of T_i and U_i, i.e.,

$$\mathbf{T} = \sum_{i=1,\cdots,d} T_i \bigotimes \left(\bigotimes_{j\neq i} U_j\right) + z \bigotimes_{i=1,\cdots,d} U_i,$$

the coefficient matrix $\mathbf{T}$ of linear system (2.3) can be diagonalized by the multi-dimensional discrete sine transform, and the diagonal element at $(\mathbf{k}, \mathbf{k})$-position is

$$\sum_{i=1,\cdots,d} T_{i,k_i} \prod_{j\neq i} U_{j,k_j} + z \prod_{i=1,\cdots,d} U_{i,k_i}.$$

Thanks to the fast algorithm for the discrete sine transform, the linear system (2.3) can be solved within $\mathcal{O}(N \ln N)$ operations with $N = \prod_{i=1}^{d} M_i$.

3. Interface Problem

We consider the following general interface problem

$$\begin{aligned}
&\mathcal{A}_{var} u \overset{def}{=} -\nabla \cdot (\beta(x)\nabla u) + c(x)u = f, && \forall\, x \in \Omega_i \cup \Omega_e,\\
&(\mathcal{D}_{\Gamma,\Omega_i} - \mathcal{D}_{\Gamma,\Omega_e})u = r, && \forall\, x \in \Gamma,\\
&\mathcal{N}_\Gamma u = m, && \forall\, x \in \Gamma,\\
&\mathcal{D}_{\partial\Omega} u = g, && \forall\, x \in \partial\Omega,
\end{aligned} \tag{3.1}$$

where $\Gamma = \partial\Omega_i \cap \partial\Omega_e$ is a Lipschitz interface, $\mathcal{N}_\Gamma = \mathcal{N}_{\Gamma,\Omega_i} + \mathcal{N}_{\Gamma,\Omega_e}$, and $\Omega = \bar{\Omega}_i \cup \Omega_e$ is a rectangular domain. See the schematic map in the left of Fig. 3.1. Here and hereafter, given any domain $\tilde{\Omega}$ and a part of its boundary $\tilde{\Gamma} \subset \partial\tilde{\Omega}$, we denote by $\mathcal{D}_{\tilde{\Gamma},\tilde{\Omega}}$ the Dirichlet operator limiting from the interior of $\tilde{\Omega}$ to its boundary. In the same spirit, we denote the Neumann operator by

$$\mathcal{N}_{\tilde{\Gamma},\tilde{\Omega}} u = \mathbf{n} \cdot \beta(x)\nabla u, \quad \forall\, x \in \tilde{\Gamma},$$

where $\mathbf{n}$ denotes the unit normal directed to the exterior of domain $\tilde{\Omega}$. In some cases, we do not specify the domain when no ambiguity occurs. The coefficient functions $\beta(x)$ and $c(x)$ are presumed continuous on each disjoint subdomain, but may be discontinuous across the interface Γ. Furthermore, we suppose $\beta(x)$ and $c(x)$ satisfy the conditions specified in (1.2).

The weak formulation associated with the interface problem (3.1) is to find a distribution $u \in L^2(\Omega)$ such that

$$\begin{aligned}
&u|_{\Omega_i} \in H^1(\Omega_i),\ u\big|_{\Omega_e} \in H^1(\Omega_e),\ \big(\mathcal{D}_{\Gamma,\Omega_i} - \mathcal{D}_{\Gamma,\Omega_e}\big)u = r \in H^{\frac{1}{2}}(\Gamma),\ \mathcal{D}_{\partial\Omega} u = g \in H^{\frac{1}{2}}(\partial\Omega),\\
&a_{var,\Omega}(u,v) \overset{def}{=} (\beta(x)\nabla u, \nabla v)_\Omega + (c(x)u, v)_\Omega = (f,v)_\Omega + < m, \mathcal{D}_\Gamma v >_\Gamma,\ \forall\, v \in H_0^1(\Omega).
\end{aligned} \tag{3.2}$$

If $f \in L^2(\Omega)$ and $m \in H^{\frac{1}{2}}(\Gamma)$, this problem is uniquely solvable (see [1]).

We resort to the conforming finite element method to solve this interface problem numerically. First we generate a uniform rectangular mesh in Ω, and then form an interface domain by collecting all rectangular elements intersecting with Γ, and make a local body-fitted *shape-regular* triangular mesh. See the schematic map in the right of Fig. 3.1.

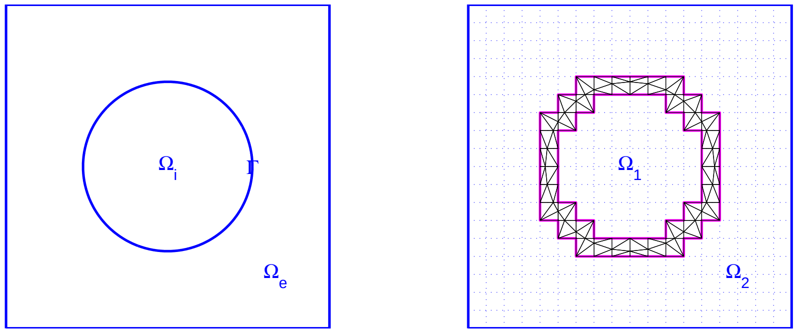

Fig. 3.1. Domain decomposition based on a uniform rectangular mesh.

The whole domain Ω is divided into three parts: Ω_1 in Ω_i, Ω_2 in Ω_e and the residual *crack* domain $\Omega_c = \text{Interior}(\Omega\backslash(\Omega_1 \cup \Omega_2))$. Note that Ω_1 and Ω_2 are composed only of rectangular elements. Set $\Omega_{i,c} = \Omega_i \cap \Omega_c$ and $\Omega_{e,c} = \Omega_e \cap \Omega_c$. The discrete variational problem corresponding to (3.2) is to find a function u such that

$$\begin{aligned}
&u|_{\Omega_1\cup\Omega_2} \in V_h^R(\Omega_1 \cup \Omega_2), \quad u|_{\Omega_{i,c}} \in V_h^T(\Omega_{i,c}), \quad u|_{\Omega_{e,c}} \in V_h^T(\Omega_{e,c}),\\
&(\mathcal{D}_{\Gamma,\Omega_{i,c}} - \mathcal{D}_{\Gamma,\Omega_{e,c}})u = I_h r, \quad \mathcal{D}_{\partial\Omega} u = I_h g,\\
&(\mathcal{D}_{\partial\Omega_c,\Omega_c} - \mathcal{D}_{\partial\Omega_c,\Omega_1\cup\Omega_2})u = 0,\\
&a_{var,\Omega}(u,v) = (f,v)_\Omega + < m, \mathcal{D}_\Gamma v >_\Gamma, \quad \forall v \in V_{h,0}(\Omega).
\end{aligned} \tag{3.3}$$

Here $V_{h,0}(\Omega)$ denotes the conforming finite element space with trace zero, which relates to the composite mesh consisting of rectangular elements in $\Omega_1 \cup \Omega_2$ and triangular elements in Ω_c. V_h^T denotes the triangular finite element space and I_h a suitable interpolating operator. We should remark that in the discrete level, the interface Γ is actually approximated with a polygon or a polytope, and the domains Ω_i and Ω_e are modified correspondingly. We hope this wilful neglect will not present any confusion to understand the main idea.

We take two steps to compute the numerical solution.

3.1. Step A: local solution

Using local triangular elements, we compute the solution of the local interface problem

$$\begin{aligned}
&\mathcal{A}_{var} u_0 = f\chi_{\Omega_c}, && x \in \Omega_c\backslash\Gamma,\\
&(\mathcal{D}_{\Gamma,\Omega_{i,c}} - \mathcal{D}_{\Gamma,\Omega_{e,c}})u_0 = r, && \forall x \in \Gamma,\\
&\mathcal{N}_\Gamma u_0 = m, && \forall x \in \Gamma,\\
&\mathcal{D}_{\partial\Omega_c,\Omega_c} u_0 = 0, && \forall x \in \partial\Omega_c.
\end{aligned}$$

Numerically, we determine u_0 such that

$$
\begin{aligned}
&u_0|_{\Omega_{i,c}} \in V_h^T(\Omega_{i,c}), \quad u_0|_{\Omega_{e,c}} \in V_h^T(\Omega_{e,c}),\\
&(\mathcal{D}_{\Gamma,\Omega_{i,c}} - \mathcal{D}_{\Gamma,\Omega_{e,c}})u_0 = I_h r, \quad \mathcal{D}_{\partial\Omega_c,\Omega_c} u_0 = 0,\\
&a_{var,\Omega_c}(u_0, v) = (f\chi_{\Omega_c}, v)_{\Omega_c} + < m, \mathcal{D}_\Gamma v >_\Gamma, \quad \forall\, v \in V_{h,0}^T(\Omega_c).
\end{aligned}
$$

Here $V_{h,0}^T$ denotes the maximal subspace of V_h^T with trace zero on $\partial\Omega_c$.

The above problem can be decomposed into three subproblems. First, using triangular elements we solve

$$
\begin{aligned}
&\mathcal{A}_{var} u_{01,i} = f\chi_{\Omega_{i,c}}, && \forall\, x \in \Omega_{i,c},\\
&\mathcal{D}_{\Gamma,\Omega_{i,c}} u_{01,i} = r, && \forall\, x \in \Gamma,\\
&\mathcal{D}_{\partial\Omega_1,\Omega_{i,c}} u_{01,i} = 0, && \forall\, x \in \partial\Omega_1,
\end{aligned}
$$

which is equivalent to determine $u_{01,i} \in V_h^T(\Omega_{i,c})$ such that

$$
\begin{aligned}
&\mathcal{D}_{\Gamma,\Omega_{i,c}} u_{01,i} = I_h r, \quad \mathcal{D}_{\partial\Omega_1,\Omega_{i,c}} u_{01,i} = 0,\\
&a_{var,\Omega_{i,c}}(u_{01,i}, v) = (f\chi_{\Omega_{i,c}}, v)_{\Omega_{i,c}}, \quad \forall\, v \in V_{h,0}^T(\Omega_{i,c}).
\end{aligned} \tag{3.4}
$$

Second, using triangular elements we solve

$$
\begin{aligned}
&\mathcal{A}_{var} u_{01,e} = f\chi_{\Omega_{e,c}}, && \forall\, x \in \Omega_{e,c},\\
&\mathcal{D}_{\Gamma,\Omega_{e,c}} u_{01,e} = 0, && \forall\, x \in \Gamma,\\
&\mathcal{D}_{\partial\Omega_2\backslash\partial\Omega,\Omega_{e,c}} u_{01,e} = 0, && \forall\, x \in \partial\Omega_2\backslash\partial\Omega,
\end{aligned}
$$

which is equivalent to determine $u_{01,e} \in V_{h,0}^T(\Omega_{e,c})$ such that

$$
a_{var,\Omega_{e,c}}(u_{01,e}, v) = (f\chi_{\Omega_{e,c}}, v)_{\Omega_{e,c}}, \quad \forall\, v \in V_{h,0}^T(\Omega_{e,c}). \tag{3.5}
$$

Let us define

$$
u_{01} = \begin{cases} u_{01,i}, & x \in \Omega_{i,c},\\ u_{01,e}, & x \in \Omega_{e,c}, \end{cases}
$$

and set $u_{02} = u_0 - u_{01}$. Then u_{02} solves

$$
\begin{aligned}
&\mathcal{A}_{var} u_{02} = 0, && \forall\, x \in \Omega_c\backslash\Gamma,\\
&(\mathcal{D}_{\Gamma,\Omega_{i,c}} - \mathcal{D}_{\Gamma,\Omega_{e,c}}) u_{02} = 0, && \forall\, x \in \Gamma,\\
&\mathcal{N}_\Gamma u_{02} = m - \mathcal{N}_\Gamma u_{01}, && \forall\, x \in \Gamma,\\
&\mathcal{D}_{\partial\Omega_c,\Omega_c} u_{02} = 0, && \forall\, x \in \partial\Omega_c,
\end{aligned}
$$

which is equivalent to determine $u_{02} \in V_{h,0}^T(\Omega_c)$ such that

$$
a_{var,\Omega_c}(u_{02}, v) =< m - \mathcal{N}_\Gamma u_{01}, v >_\Gamma, \quad \forall\, v \in V_{h,0}^T(\Omega_c). \tag{3.6}
$$

In the case that all interior vertices of Ω_c are located in the interface Γ, there is no need to solve (3.4)-(3.5), since the solutions are simply the piecewise linear interpolants. The subproblem (3.6) can be solved efficiently, since the resulting linear system only involves the degrees of freedom on the interface Γ, and the coefficient matrix is sparse and symmetric positive definite. Especially in two dimensions, if the interface is topologically isomorphic to a circle, this coefficient matrix can be made cyclic tridiagonal, and the linear system can be solved directly within the linear scaling complexity.

3.2. Step B: global solution

Let us extend the function u_0 to the whole domain Ω by zero and put $u = u_0 + \tilde{u}$. It is easy to verify that $\tilde{u}$ solves the following interface problem

$$\begin{aligned}
&\mathcal{A}_{var}\tilde{u} = f\chi_{\Omega_1\cup\Omega_2}, && \forall\, x \in \Omega_1 \cup \Omega_2 \cup \Omega_c,\\
&(\mathcal{D}_{\partial\Omega_c,\Omega_c} - \mathcal{D}_{\partial\Omega_c,\Omega_1\cup\Omega_2})\tilde{u} = 0, && \forall\, x \in \partial\Omega_c,\\
&\mathcal{N}_{\partial\Omega_c}\tilde{u} = -\mathcal{N}_{\partial\Omega_c}u_0, && \forall\, x \in \partial\Omega_c,\\
&\mathcal{D}_{\partial\Omega}\tilde{u} = g, && \forall\, x \in \partial\Omega.
\end{aligned}$$

Numerically, $\tilde{u} \in V_h(\Omega)$ satisfies $\mathcal{D}_{\partial\Omega}\tilde{u} = I_h g$ and

$$a_{var,\Omega}(\tilde{u}, v) = (f\chi_{\Omega_1\cup\Omega_2}, v)_{\Omega_1\cup\Omega_2} - < \mathcal{N}_{\partial\Omega_c}u_0, v >_{\partial\Omega_c}, \quad \forall\, v \in V_{h,0}(\Omega). \tag{3.7}$$

Let us denote by $\mathcal{S}_{loc}^{T(R)} : t \to u_t$ the solution operator of the local Dirichlet variational problem with *Triangular* (*Rectangular*) elements, i.e.,

$$u_t \in V_h^{T(R)}(\Omega_c),\ u_t|_{\partial\Omega_c} = t,\ a_{var,\Omega_c}(u_t, v) = 0, \quad \forall\, v \in V_{h,0}^{T(R)}(\Omega_c), \tag{3.8}$$

and $\mathcal{K}_{var}^{T(R)}$ the corresponding discrete Dirichlet-to-Neumann map

$$< \mathcal{K}_{var}^{T(R)}(t), \mathcal{D}_{\partial\Omega_c}v >_{\partial\Omega_c} = a_{var,\Omega_c}(u_t, v), \quad \forall\, v \in V_h^{T(R)}(\Omega_c).$$

Recalling (3.7) we have

$$a_{var,\Omega_c}(\tilde{u}, v) = 0, \quad \forall\, v \in V_{h,0}^{T}(\Omega_c).$$

Therefore, for any $v \in V_{h,0}(\Omega)$ it holds

$$\begin{aligned}
a_{var,\Omega}(\tilde{u}, v) &= a_{var,\Omega_1\cup\Omega_2}(\tilde{u}, v) + a_{var,\Omega_c}(\tilde{u}, v)\\
&= a_{var,\Omega_1\cup\Omega_2}(\tilde{u}, v) + < \mathcal{K}_{var}^{T}\mathcal{D}_{\partial\Omega_c}\tilde{u}, \mathcal{D}_{\partial\Omega_c}v >_{\partial\Omega_c}.
\end{aligned}$$

Thus then, the variational problem (3.7) is equivalent to find $\tilde{u} \in V_h^R(\Omega_1 \cup \Omega_2)$, such that $\mathcal{D}_{\partial\Omega}\tilde{u} = I_h g$ and

$$\begin{aligned}
&a_{var,\Omega_1\cup\Omega_2}(\tilde{u}, v) + < \mathcal{K}_{var}^{T}\mathcal{D}_{\partial\Omega_c}\tilde{u}, \mathcal{D}_{\partial\Omega_c}v >_{\partial\Omega_c}\\
&= (f\chi_{\Omega_1\cup\Omega_2}, v)_{\Omega_1\cup\Omega_2} - < \mathcal{N}_{\partial\Omega_c}u_0, v >_{\partial\Omega_c}, \quad \forall\, v \in V_{h,*}^R(\Omega_1 \cup \Omega_2),
\end{aligned} \tag{3.9}$$

where $V_{h,*}^R(\Omega_1 \cup \Omega_2)$ is the maximal subspace of $V_h^R(\Omega_1 \cup \Omega_2)$ with zero trace on $\partial\Omega$.

Lemma 3.1. *There exist two positive constants c_1 and c_2, independent of h, such that*

$$c_1 \le \frac{< \mathcal{K}_{var}^R(t), t >_{\partial\Omega_c}}{< \mathcal{K}_{var}^T(t), t >_{\partial\Omega_c}} \le c_2, \quad \forall\, 0 \ne t \in \partial V_h^T(\Omega_c) = \partial V_h^R(\Omega_c).$$

This lemma is a direct consequence of the discrete harmonic extension theorem, see Theorem 11.4.3 in [12].

Theorem 3.1. *For any $z \ge 0$, let $\mathcal{K}_z^R$ be the discrete Dirichlet-to-Neumann operator of Poisson operator $-\Delta + zI$ in Ω_c with rectangular elements. Then the bilinear form*

$$a_{z,\Omega_1\cup\Omega_2}(u, v) + < \mathcal{K}_z^R\mathcal{D}_{\partial\Omega_c}u, \mathcal{D}_{\partial\Omega_c}v >_{\partial\Omega_c}$$

defines a uniformly equivalent inner product as

$$a_{var,\Omega_1\cup\Omega_2}\big(u,v\big)+<\mathcal{K}_{var}^T\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}v>_{\partial\Omega_c}$$

in $V_{h,*}^R(\Omega_1\cup\Omega_2)$*. Precisely, there exist two positive constants* c_3 *and* c_4*, independent of* h*, such that*

$$c_3\le\frac{a_{var,\Omega_1\cup\Omega_2}\big(u,u\big)+<\mathcal{K}_{var}^T\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}}{a_{z,\Omega_1\cup\Omega_2}\big(u,u\big)+<\mathcal{K}_{z}^R\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}}\le c_4,\quad\forall\,0\ne u\in V_{h,*}^R(\Omega_1\cup\Omega_2).$$

Proof. Since

$$\begin{aligned}&a_{z,\Omega_1\cup\Omega_2}(u,u)+<\mathcal{K}_{z}^R\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}\\ \le&\max(1,z)\Big(a_{1,\Omega_1\cup\Omega_2}(u,u)+<\mathcal{K}_{1}^R\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}\Big),\end{aligned}$$

and

$$\begin{aligned}&a_{z,\Omega_1\cup\Omega_2}(u,u)+<\mathcal{K}_{z}^R\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}\\ \ge&a_{0,\Omega_1\cup\Omega_2}(u,u)+<\mathcal{K}_{0}^R\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c},\end{aligned}$$

it suffices to show that there exist two positive constants c_5 and c_6, independent of h, such that

$$\frac{a_{var,\Omega_1\cup\Omega_2}\big(u,u\big)+<\mathcal{K}_{var}^T\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}}{a_{0,\Omega_1\cup\Omega_2}\big(u,u\big)+<\mathcal{K}_{0}^R\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}}\le c_5,\tag{3.10}$$

$$\frac{a_{var,\Omega_1\cup\Omega_2}\big(u,u\big)+<\mathcal{K}_{var}^T\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}}{a_{1,\Omega_1\cup\Omega_2}\big(u,u\big)+<\mathcal{K}_{1}^R\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}}\ge c_6.\tag{3.11}$$

Let $S_{loc,z}^{T(R)}$ be the discrete solution operator of Poisson operator $-\Delta+zI$ with *Triangular* and *Rectangular* elements respectively. By the Dirichlet principle, we have

$$<\mathcal{K}_{var}^T\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}\le a_{var,\Omega_c}\Big(S_{loc,0}^T\mathcal{D}_{\partial\Omega_c}u,S_{loc,0}^T\mathcal{D}_{\partial\Omega_c}u\Big).$$

Thus then, there exists a positive constant c_7 such that

$$\begin{aligned}&a_{var,\Omega_1\cup\Omega_2}\big(u,u\big)+<\mathcal{K}_{var}^T\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}\\ \le&\ a_{var,\Omega_1\cup\Omega_2}\big(u,u\big)+a_{var,\Omega_c}\big(S_{loc,0}^T\mathcal{D}_{\partial\Omega_c}u,S_{loc,0}^T\mathcal{D}_{\partial\Omega_c}u\big)\\ \le&\ c_7\Big(a_{0,\Omega_1\cup\Omega_2}\big(u,u\big)+a_{0,\Omega_c}\big(S_{loc,0}^T\mathcal{D}_{\partial\Omega_c}u,S_{loc,0}^T\mathcal{D}_{\partial\Omega_c}u\big)\Big).\end{aligned}\tag{3.12}$$

The last inequality holds since $a_{0,\Omega}$ is an equivalent bilinear form as $a_{var,\Omega}$ in $H_0^1(\Omega)$. By Lemma 3.1, there exists a positive constant c_8 such that

$$a_{0,\Omega_c}(S_{loc,0}^T\mathcal{D}_{\partial\Omega_c}u,S_{loc,0}^T\mathcal{D}_{\partial\Omega_c}u)\le c_8a_{0,\Omega_c}(S_{loc,0}^R\mathcal{D}_{\partial\Omega_c}u,S_{loc,0}^R\mathcal{D}_{\partial\Omega_c}u).\tag{3.13}$$

Combining (3.12)-(3.13) then yields (3.10). The proof of (3.11) is analogous. By the Dirichlet principle, we have

$$<\mathcal{K}_{1}^R\mathcal{D}_{\partial\Omega_c}u,\mathcal{D}_{\partial\Omega_c}u>_{\partial\Omega_c}\le a_{1,\Omega_c}(S_{loc}^R\mathcal{D}_{\partial\Omega_c}u,S_{loc}^R\mathcal{D}_{\partial\Omega_c}u).$$

Thus then, there exists a positive constant c_9 such that

$$\begin{aligned}
&a_{1,\Omega_1\cup\Omega_2}(u,u)+ < \mathcal{K}_1^R \mathcal{D}_{\partial\Omega_c} u, \mathcal{D}_{\partial\Omega_c} u >_{\partial\Omega_c} \\
\le\ & a_{1,\Omega_1\cup\Omega_2}(u,u) + a_{1,\Omega_c}(S_{loc}^R \mathcal{D}_{\partial\Omega_c} u, S_{loc}^R \mathcal{D}_{\partial\Omega_c} u) \\
\le\ & c_9 \left(a_{var,\Omega_1\cup\Omega_2}(u,u) + a_{var,\Omega_c}(S_{loc}^R \mathcal{D}_{\partial\Omega_c} u, S_{loc}^R \mathcal{D}_{\partial\Omega_c} u) \right).
\end{aligned} \tag{3.14}$$

The last inequality holds since $a_{1,\Omega}$ is an equivalent bilinear form as $a_{var,\Omega}$ in $H_0^1(\Omega)$. By Lemma 3.1, there exists a positive constant c_{10} such that

$$a_{var,\Omega_c}(S_{loc}^R \mathcal{D}_{\partial\Omega_c} u, S_{loc}^R \mathcal{D}_{\partial\Omega_c} u) \le c_{10} a_{var,\Omega_c}(S_{loc}^T \mathcal{D}_{\partial\Omega_c} u, S_{loc}^T \mathcal{D}_{\partial\Omega_c} u). \tag{3.15}$$

Combining (3.14)-(3.15) then yields (3.11). □

The solution of (3.9) is the Riesz representation of the right hand functional with respect to the inner product

$$a_{var,\Omega_1\cup\Omega_2}(u,v)+ < \mathcal{K}_{var}^T \mathcal{D}_{\partial\Omega_c} u, \mathcal{D}_{\partial\Omega_c} v >_{\partial\Omega_c}.$$

Theorem 3.1 implies that we can precondition (3.9) by the Poisson equation with rectangular elements. A CG algorithm can be developed as described in Section 2. The convergence speed is independent of the mesh size and the interface geometry. At each iteration, one needs to solve a local Dirichlet boundary value problem and a Dirichlet Poisson problem. The computational complexity is essentially of linear scaling with respect to the total number of degrees of freedom.

4. Neumann Boundary Value Problem

In this section, we consider the following Neumann boundary value problem

$$\begin{aligned}
&\mathcal{A}_{var} u \overset{def}{=} -\nabla\cdot(\beta(x)\nabla u) + c(x)u = f, \quad \forall\, x \in \Omega_i, \\
&\mathcal{N}_{\Gamma,\Omega_i} u = m, \quad \forall\, x \in \Gamma,
\end{aligned} \tag{4.1}$$

where $\Gamma = \partial\Omega_i$ is a Lipschitz boundary. The variational problem associated with (4.1) is to find $u \in H^1(\Omega_i)$ such that

$$a_{var,\Omega_i}(u,v) = (f,v)_{\Omega_i} + < m, \mathcal{D}_\Gamma v >_\Gamma, \quad \forall\, v \in H^1(\Omega_i). \tag{4.2}$$

Suppose $\beta(x)$ and $c(x)$ are specified such that $a_{var,\Omega_i}(\cdot,\cdot)$ defines an equivalent inner product as $a_{1,\Omega_i}(\cdot,\cdot)$ in $H^1(\Omega_i)$, the variational problem (4.2) is uniquely solvable for any $f \in L^2(\Omega_i)$ and $m \in H^{-\frac{1}{2}}(\Gamma)$.

We first embed the domain Ω_i into a large rectangular domain Ω. See the schematic map in the left of Fig. 3.1. Applying the method described in Section 3, we derive a composite shape-regular mesh in Ω_i. The discrete variational problem is then to find $u \in V_h(\Omega_i)$ such that

$$a_{var,\Omega_i}(u,v) = (f,v)_{\Omega_i} + < m, \mathcal{D}_\Gamma v >_\Gamma, \quad \forall\, v \in V_h(\Omega_i). \tag{4.3}$$

We take two steps to compute the solution of (4.3).

Algorithm 4.1.

1. Solve the following local boundary value problem

$$\begin{aligned} &\mathcal{A}_{var} u_0 = f, && \forall\, x \in \Omega_{i,c}, \\ &\mathcal{D}_{\partial\Omega_1} u_0 = 0, && \forall\, x \in \partial\Omega_1, \\ &\mathcal{N}_{\Gamma,\Omega_{i,c}} u_0 = m, && \forall\, x \in \Gamma. \end{aligned} \tag{4.4}$$

Set
$$V_{h,*}^T(\Omega_{i,c}) = \Big\{ v \in V_h^T(\Omega_{i,c}) : \mathcal{D}_{\partial\Omega_1} v = 0 \Big\}.$$

The variational form of (4.4) is to find $u_0 \in V_{h,*}^T(\Omega_{i,c})$ such that

$$a_{var,\Omega_{i,c}}(u_0, v) = (f, v)_{\Omega_{i,c}} + < m, \mathcal{D}_\Gamma v >_\Gamma, \ \forall\, v \in V_{h,*}^T(\Omega_{i,c}).$$

This problem can be solved efficiently, since only a few degrees of freedom around the boundary Γ get involved.

2. Set $\tilde{u} = u - u_0$, and determine $\tilde{u}$ by solving the following interface problem

$$\begin{aligned} &\mathcal{A}_{var} \tilde{u} = f, && \forall\, x \in \Omega_1, \\ &\mathcal{A}_{var} \tilde{u} = 0, && \forall\, x \in \Omega_{i,c}, \\ &(\mathcal{D}_{\partial\Omega_1,\Omega_1} - \mathcal{D}_{\partial\Omega_1,\Omega_{i,c}})\tilde{u} = 0, && \forall\, x \in \partial\Omega_1, \\ &\mathcal{N}_{\partial\Omega_1} \tilde{u} = -\mathcal{N}_{\partial\Omega_1,\Omega_{i,c}} u_0, && \forall\, x \in \partial\Omega_1, \\ &\mathcal{N}_{\Gamma,\Omega_{i,c}} \tilde{u} = 0, && \forall\, x \in \Gamma. \end{aligned} \tag{4.5}$$

Let $\mathcal{S}_{var}^T : r \to u_r$ be the discrete solution operator of the boundary value problem

$$\begin{aligned} &\mathcal{A}_{var} u_r = 0, && \forall\, x \in \Omega_{i,c}, \\ &\mathcal{N}_{\Gamma,\Omega_{i,c}} u_r = 0, && \forall\, x \in \Gamma, \\ &\mathcal{D}_{\partial\Omega_1,\Omega_{i,c}} u_r = r, && \forall\, x \in \partial\Omega_1, \end{aligned} \tag{4.6}$$

and let $\mathcal{K}_{var}^T$ denote the discrete Dirichlet-to-Neumann map, i.e.,

$$< \mathcal{K}_{var}^T(r), \mathcal{D}_{\partial\Omega_1} v >_{\partial\Omega_1} = a_{var,\Omega_1}(u_r, v), \quad \forall\, v \in V_h^T(\Omega_{i,c}).$$

Then confined to Ω_1, $\tilde{u}$ solves the variational problem: find $\tilde{u} \in V_h^R(\Omega_1)$ such that

$$\begin{aligned} &a_{var,\Omega_1}(\tilde{u}, v) + < \mathcal{K}_{var}^T \mathcal{D}_{\partial\Omega_1} \tilde{u}, \mathcal{D}_{\partial\Omega_1} v >_{\partial\Omega_1} \\ &= < f, v >_{\Omega_1} - < \mathcal{N}_{\partial\Omega_1,\Omega_{i,c}} u_0, \mathcal{D}_{\partial\Omega_1} v >_{\partial\Omega_1}, \quad \forall\, v \in V_h^R(\Omega_1). \end{aligned} \tag{4.7}$$

Let $\mathcal{S}_z^R : r \in \partial V_h^R(\Omega_1) \to u_r \in V_h^R(\Omega_c \cup \Omega_2)$ be the discrete solution operator with rectangular elements of the following problem

$$\begin{aligned} &u_r \in V_h^R(\Omega_c \cup \Omega_2), \ u_r|_{\partial\Omega} = 0, \\ &a_{z,\Omega_c\cup\Omega_2}(u_r, v) = 0, \quad \forall\, v \in V_h^R(\Omega_c \cup \Omega_2) \text{ with } v|_{\partial\Omega} = 0, \end{aligned} \tag{4.8}$$

and $\mathcal{K}_z^R$ the corresponding discrete Dirichlet-to-Neumann map defined by

$$< \mathcal{K}_z^R(r), \mathcal{D}_{\partial\Omega_1} v >_{\partial\Omega_1} = a_{z,\Omega_c\cup\Omega_2}(u_r, v), \quad \forall\, v \in V_h^R(\Omega_c \cup \Omega_2) \text{ with } v|_{\partial\Omega} = 0.$$

Theorem 4.1. *The bilinear form*

$$a_{z,\Omega_1}(u, v) + < \mathcal{K}_z^R \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} v >_{\partial\Omega_1}$$

defines a uniformly equivalent inner product as

$$a_{var,\Omega_1}(u, v) + < \mathcal{K}_{var}^T \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} v >_{\partial\Omega_1}$$

in $V_h^R(\Omega_1)$.

Proof. Let $\mathcal{S}_z : r \in \partial V_h^R(\Omega_1) \to u_r \in V_h(\Omega_c \cup \Omega_2)$ be the discrete solution operator with rectangular elements of the following problem

$$\begin{aligned} & u_r \in V_h(\Omega_c \cup \Omega_2),\ u_r|_{\partial\Omega} = 0, \\ & a_{z,\Omega_c\cup\Omega_2}(u_r, v) = 0, \quad \forall\, v \in V_h(\Omega_c \cup \Omega_2) \text{ with } v|_{\partial\Omega} = 0, \end{aligned} \tag{4.9}$$

and $\mathcal{K}_z$ the corresponding discrete Dirichlet-to-Neumann map defined by

$$< \mathcal{K}_z(r), \mathcal{D}_{\partial\Omega_1} v >_{\partial\Omega_1} = a_{z,\Omega_c\cup\Omega_2}(u_r, v), \quad \forall\, v \in V_h(\Omega_c \cup \Omega_2) \text{ with } v|_{\partial\Omega} = 0.$$

By the Dirichlet principle, we have

$$\begin{aligned} < \mathcal{K}_{var}^T \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1} &= a_{var,\Omega_i\cap\Omega_c}(\mathcal{S}_{var}^T \mathcal{D}_{\partial\Omega_1} u, \mathcal{S}_{var}^T \mathcal{D}_{\partial\Omega_1} u) \\ &\le a_{var,\Omega_i\cap\Omega_c}(\mathcal{S}_z \mathcal{D}_{\partial\Omega_1} u, \mathcal{S}_z \mathcal{D}_{\partial\Omega_1} u). \end{aligned}$$

Thus then,

$$\begin{aligned} & a_{var,\Omega_1}(u, u) + < \mathcal{K}_{var}^T \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1} \\ \le\ & a_{var,\Omega_1}(u, u) + a_{var,\Omega_i\cap\Omega_c}(\mathcal{S}_z \mathcal{D}_{\partial\Omega_1} u, \mathcal{S}_z \mathcal{D}_{\partial\Omega_1} u) \\ \le\ & \max(\beta_1, c_{max})\Big(a_{1,\Omega_1}(u, u) + a_{1,\Omega_i\cap\Omega_c}(\mathcal{S}_z \mathcal{D}_{\partial\Omega_1} u, \mathcal{S}_z \mathcal{D}_{\partial\Omega_1} u)\Big) \\ \le\ & \max(\beta_1, c_{max})\Big(a_{1,\Omega_1}(u, u) + a_{1,\Omega_c\cup\Omega_2}(\mathcal{S}_z \mathcal{D}_{\partial\Omega_1} u, \mathcal{S}_z \mathcal{D}_{\partial\Omega_1} u)\Big) \\ \le\ & c_1 \max(\beta_1, c_{max})\Big(a_{z,\Omega_1}(u, u) + a_{z,\Omega_c\cup\Omega_2}(\mathcal{S}_z \mathcal{D}_{\partial\Omega_1} u, \mathcal{S}_z \mathcal{D}_{\partial\Omega_1} u)\Big) \\ =\ & c_1 \max(\beta_1, c_{max})\Big(a_{z,\Omega_1}(u, u) + < \mathcal{K}_z \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1}\Big). \end{aligned}$$

The last second inequality holds since $a_{var,\Omega}(\cdot,\cdot)$ is equivalent to $a_{1,\Omega}(\cdot,\cdot)$ in $H_0^1(\Omega)$. Recalling Lemma 3.1 we know there exists $c_2 > 0$ such that

$$\frac{< \mathcal{K}_z \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1}}{< \mathcal{K}_z^R \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1}} \le c_2.$$

Therefore,

$$\begin{aligned} & a_{var,\Omega_1}(u, u) + < \mathcal{K}_{var}^T \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1} \\ \le\ & c_1 \max(1, c_2) \max(\beta_1, c_{max}) \left(a_{z,\Omega_1}(u, u) + < \mathcal{K}_z^R \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1}\right). \end{aligned}$$

On the other hand, since $\partial\Omega_1$ is uniformly Lipschitzian and $\mathcal{S}_z^R$ is a discrete Poisson extension operator, there exists a constant $c_3 > 0$ independent of h, such that

$$< \mathcal{K}_z^R \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1} = a_{1,\Omega_c\cup\Omega_2}(\mathcal{S}_z^R \mathcal{D}_{\partial\Omega_1} u, \mathcal{S}_z^R \mathcal{D}_{\partial\Omega_1} u) \leq c_3 \|\mathcal{D}_{\partial\Omega_1} u\|^2_{H^{\frac{1}{2}}(\partial\Omega_1)}.$$

By the trace theorem, there exists a constant $c_4 > 0$ independent of h such that

$$\|\mathcal{D}_{\partial\Omega_1} u\|^2_{H^{\frac{1}{2}}(\partial\Omega_1)} \leq c_4 a_{1,\Omega_1}(u,u).$$

Thus then,

$$a_{z,\Omega_1}(u,u) + < \mathcal{K}_z^R \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1} \leq (c_3 c_4 + \max(1,z)) a_{1,\Omega_1}(u,u).$$

By the assumption that $a_{var,\Omega_1}(\cdot,\cdot)$ defines an equivalent inner product as $a_{1,\Omega_1}(\cdot,\cdot)$, there exist a constant $c_5 > 0$ such that

$$\begin{aligned}
& a_{z,\Omega_1}(u,u) + < \mathcal{K}_z^R \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1} \\
\leq & c_5 \Big(c_3 c_4 + \max(1,z) \Big) a_{var,\Omega_1}(u,u) \\
\leq & c_5 \Big(c_3 c_4 + \max(1,z) \Big) \Big(a_{var,\Omega_1}(u,u) + < \mathcal{K}_{var}^T \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} u >_{\partial\Omega_1} \Big).
\end{aligned}$$

This ends the proof. □

Theorem 4.1 implies that one can precondition (4.7) with the inner product

$$a_{z,\Omega_1}(u,v) + < \mathcal{K}_z^R \mathcal{D}_{\partial\Omega_1} u, \mathcal{D}_{\partial\Omega_1} v >_{\partial\Omega_1} .$$

The Riesz representation with respect to this inner product corresponds to resolving a Dirichlet Poisson problem in the rectangular domain Ω. The CG method described in Section 2 can then be applied to solve (4.7) efficiently.

5. Numerical Experiments

We have chosen four simple problems to validate the performance of the proposed algorithms. In particular, we are interested in the convergence speed of CG method under different mesh sizes. In all numerical examples, we set the relative residual error tolerance as 10^{-6}, and the preconditioning Poisson operator as $-\Delta$, that is to say, we set $z = 0$.

All examples are two-dimensional and computed on some square domain with equally spaced nodes. In the following, *Node Number* stands for the degrees of freedom in one direction. Thus a numerical test with a node number of 1024 implies that the total number of grid points is around one million.

5.1. Example 1

We consider an interface problem of Laplace equation in the rectangular domain $[-5,5]^2$. The interface is a circle with radius $R = \pi$. The exact solution is set as

$$u(x) = \begin{cases} 1 & , |x| < R, \\ 1 - \log \frac{|x|}{R} & , |x| > R. \end{cases}$$

It is easy to verify that the Dirichlet jump at the interface is $r = 0$, and the Neumann jump is $m = 1/R$, see (3.1).

The left of Fig. 5.1 plots the numerical solution when the node number is equal to 64, while the right shows the convergence rates in both L_∞ and L_2 norm. Second order accuracy can be observed. Besides, the CG iteration number remains stable when the mesh size gets smaller. This coincides with our theoretical analysis given in Section 3 that the convergence speed of CG method is uniform with respect to the mesh size.

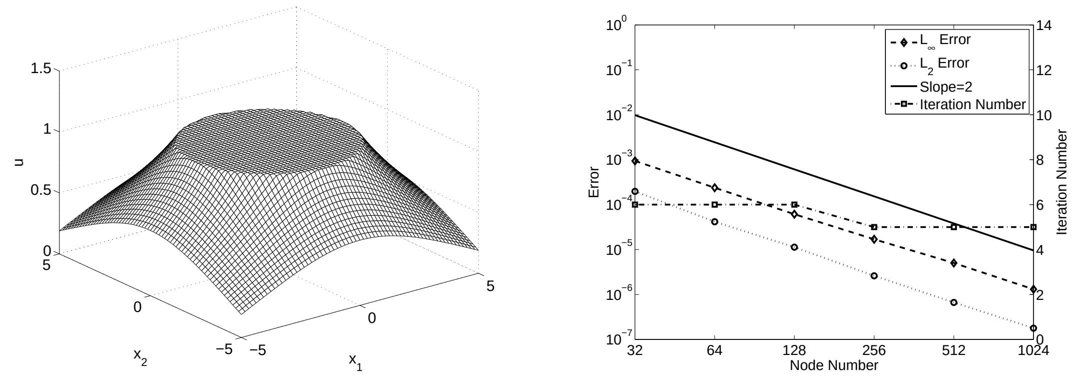

Fig. 5.1. Example 1. Left: numerical solution with a node number of 64. Right: errors and CG iteration number vs the node number.

5.2. Example 2

In this example, we consider an interface problem with variable coefficients. The computational domain is $[-\pi/3, \pi/3]^2$, and the interface is a circle with radius 0.5. We set the exact solution as

$$u(x) = \begin{cases} e^{x_1}\cos x_2, & |x| < 0.5, \\ 0, & |x| > 0.5, \end{cases}$$

and the diffusion coefficient as

$$\beta(x) = \begin{cases} |x|^2 + 1, & |x| < 0.5, \\ 1, & |x| > 0.5. \end{cases}$$

A direct computation shows that the source term is

$$f(x) = \begin{cases} 2e^{x_1}(x_2 \sin x_2 - x_1 \cos x_2), & |x| < 0.5, \\ 0, & |x| > 0.5, \end{cases}$$

and the jump conditions are

$$r = e^{x_1}\cos x_2, \ m = \frac{5}{2}e^{x_1}(x_1 \cos x_2 - x_2 \sin x_2).$$

Fig. 5.2 shows the numerical solution and the errors. Nearly second order can be observed. The CG iteration number also remains stable when the mesh gets refined.

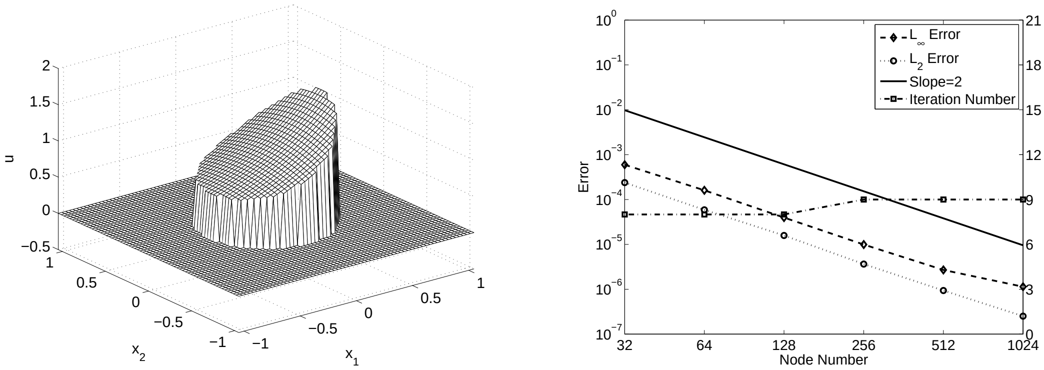

Fig. 5.2. Example 2. Left: numerical solution with a node number of 64. Right: errors and CG iteration number vs the node number.

5.3. Example 3

In this example, we check the performance of the proposed algorithm on the interface geometry and the diffusion coefficient function. As in the last example, we set the computational domain as $[-\pi/3, \pi/3]^2$. The interface Γ is parameterized and given by

$$x_1 = r(\theta)\cos\theta,\ x_2 = r(\theta)\sin\theta,\ r(\theta) = 0.5 + 0.2\sin(\omega\theta), \quad 0 \le \theta < 2\pi.$$

The diffusion coefficient is piecewise constant, i.e.,

$$\beta(x) = \begin{cases} \beta_-, & x \in \Omega_i, \\ \beta_+, & x \in \Omega_e, \end{cases}$$

and we set the exact solution as

$$u(x) = \begin{cases} \frac{r^2}{\beta_-}, & x \in \Omega_i, \\ \frac{r^4}{\beta_+}, & x \in \Omega_e. \end{cases}$$

Fig. 5.3 shows the numerical solutions under different coefficient settings. In Fig. 5.4, we plot the numerical errors and CG iteration numbers for $\beta_+ = 2, 10, 100$ respectively. The parameter β_- is fixed as 1. We could still observe that the errors degenerate with second order rate when the mesh size is sufficiently small. For $\beta_+ = 2, 10$, the iteration numbers do not change much, but increase as β increases. For $\beta_+ = 100$, the iteration number becomes as large as 100. This indicates that the condition number of preconditioned problem is still large, though

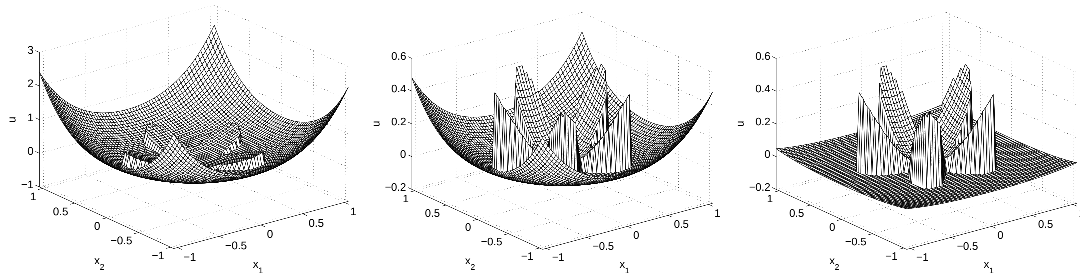

Fig. 5.3. Numerical solution u of Example 3. $\omega = 5$. Left: $\beta_- = 1$, $\beta_+ = 2$. Middle: $\beta_- = 1$, $\beta_+ = 10$. Right: $\beta_- = 1$, $\beta_+ = 100$.

Table 5.1: Example 3. Node number is set as 1024.

	$\omega=1$		$\omega=5$		$\omega=10$	
	L_∞	Ite. Num.	L_∞	Ite. Num.	L_∞	Ite. Num.
$\beta_+=2$	1.80E-6	14	3.52E-6	14	6.58E-6	14
$\beta_+=10$	9.95E-7	33	1.37E-6	32	2.08E-6	33
$\beta_+=100$	1.20E-6	96	1.21E-6	100	1.19E-6	101

independent of the mesh size. On the other hand, if we fix the coefficient function but change the interface geometry by varying ω (see Table 5.1), we see that the iteration numbers almost remain constant. These imply that our algorithm is sensitive to the contrast of coefficient functions, but not to the interface geometry. Actually, the latter point has been shown by our theoretical investigation, see Theorem 3.1.

5.4. Example 4

In this example, we consider a Neumann boundary value problem with variable coefficients

$$\begin{aligned} &-\nabla\cdot(\beta(x)\nabla u)+c(x)u=f, \quad \forall\, x\in\Omega_i=\{x:|x|\le 0.5\},\\ &\mathbf{n}\cdot\beta(x)\nabla u=m, \quad \forall\, x\in\Gamma, \end{aligned} \tag{5.1}$$

where

$$\beta(x)=2+\sin(x_1+x_2), \quad c(x)=x_1^2+x_2^2.$$

The exact solution is set as $u(x)=e^{x_1}\cos x_2$. Other data functions are set correspondingly.

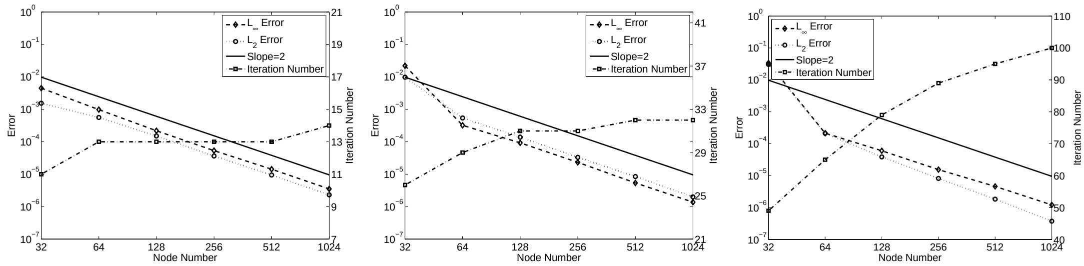

Fig. 5.4. Errors and iteration numbers of Example 3. $\omega=5$. Left: $\beta_-=1$, $\beta_+=2$. Middle: $\beta_-=1$, $\beta_+=10$. Right: $\beta_-=1$, $\beta_+=100$.

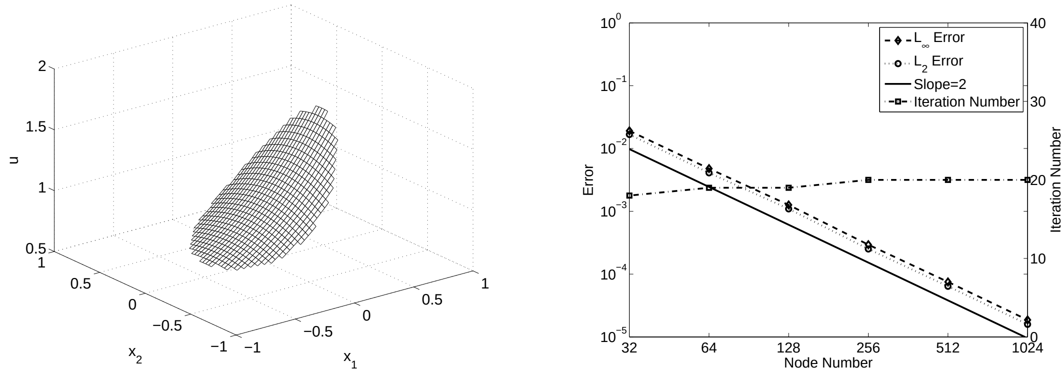

Fig. 5.5. Example 4. Left: numerical solution. Right: errors and CG iteration numbers.

We embed the circular domain into the square $[-\pi/3, \pi/3]^2$ and apply the algorithm described in Section 4. The left of Fig. 5.5 shows the numerical solution when the node number is 64. The right plot the numerical errors and the CG iteration numbers. A second order convergence rate is clearly observed in both L_∞ and L_2 norms. The CG iteration number also remains very stable as the mesh size becomes smaller.

6. Conclusion

We have developed fast algorithms for the self-adjoint interface problem and the Neumann boundary value problem. The basic idea is to precondition the PDE problem with the Poisson equation in a rectangular domain. Since the interface and the physical boundary may not be aligned with the uniform rectangular mesh, we need to reformulate the PDE problem into a suitable form such that the Poisson preconditioning is applicable. We have shown how this can be achieved and proved the associated operators of the preconditioned operator equations are self-adjoint and uniformly positive definite. This implies that the condition numbers of these operators are independent of the mesh size. Numerical examples have validated this result.

There are two important issues which are left open in this paper. First, when the fluctuation of coefficient functions is large, the condition number of preconditioned system is still relatively large, though independent of the mesh size. More efficient preconditioners are still wanting. Second, the algorithm for the Neumann boundary value problem cannot be straightforwardly adapted to the Dirichlet boundary value problem. The new difficulty lies in the fact that unlike the Neumann boundary value problem, the Dirichlet energy in the local region for the Dirichlet boundary value problem cannot be *uniformly* bounded by the Dirichlet energy in the interior region composed of rectangular elements. We have succeeded in developing a two-stage CG algorithm for the Dirichlet boundary value problem. However, since this algorithm involves a nested loop, the efficiency of this algorithm is inferior to the algorithm developed in this paper for the Neumann boundary value problem.

Acknowledgments. The work of the first author was supported by the National Natural Science Foundation of China (91330203). The work of the second author was supported by the National Natural Science Foundation of China (10371218) and the Initiative Scientific Research Program of Tsinghua University.

References

[1] Z.M. Chen and J. Zou, Finite element methods and their convergence for elliptic and parabolic interface problems, *Numer. Math.*, **79** (1998), 175-202.

[2] X. He, T. Lin and Y. Lin, Immersed finite element methods for elliptic interface problems with non-homogeneous jump conditions, *Inter. J. Numer. Anal. Modeling*, **8**:2 (2011), 284-301.

[3] S.M. Hou and X.D. Liu, A numerical method for solving variable coefficient elliptic equation with interfaces, *J. Comput. Phys.*, **202** (2005), 411-445.

[4] H. Huang and Z. Li, Convergence analysis of the immersed interface method, *IMA J. Numer. Anal.*, **19** (1999), 583-608.

[5] R.J. Leveque and Z. Li, The immersed boundary method for elliptic equations with discontinuous coefficients and singular sources, *SIAM J. Numer. Anal.*, **31**:4 (1994), 1019-1044.

[6] Z. Li, A fast iterative algorithm for elliptic interface problems, *SIAM J. Numer. Anal.*, **35**:1 (1998), 230-254.

[7] Z. Li and K. Ito, Maximum principle preserving schemes for interface problems with discontinuous coefficients, *SIAM J. Sci. Comput.*, **23** (2001), 330-361.

[8] Z. Li, T. Lin and X.H. Wu, New Cartesian grid methods for interface problems using the finite element formulation, *Numer. Math.*, **96**:1 (2003), 61-98.

[9] C.S. Peskin, Numerical analysis of blood flow in heart, *J. Comput. Phys.*, **25** (1977), 220-252.

[10] C.S. Peskin, The immersed boundary method, *Acta Numerica,* (2002), 479-517.

[11] Y. Saad, Iterative methods for sparse linear systems, Philadelphia, SIAM, 2003.

[12] L. Wang and X. Xu, Mathematical foundation of finite element methods (in chinese), Science Press, Beijing, 2004.

[13] H. Xie, Z. Li and Z. Qiao, A finite element method for elasticity interface problems with locally modified triangulations, *Inter. J. Numer. Anal. Modeling*, **8**:2 (2011), 189-200.