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HOMOTOPY CONTINUATION METHODS FOR STOCHASTIC TWO-POINT BOUNDARY VALUE PROBLEMS DRIVEN BY ADDITIVE NOISES*

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Abstract

In this paper the homotopy continuation method for stochastic two-point boundary value problems driven by additive noises is studied. The existence of the solution of the homotopy equation is proved. Numerical schemes are constructed and error estimates are obtained. Numerical experiments demonstrate the effectiveness of the homotopy continuation method over other commonly used methods such as the shooting method.

Mathematics subject classification: 65L10, 65L12, 65C30.
Key words: Stochastic differential equations, Homotopy method, Shooting method.

1. Introduction

In recent years much progress has been made in numerical methods for initial value problems of stochastic differential equations and stochastic partial differential equations see, e.g., [10,13,19,23,25,27,28]. In contrast, numerical methods for stochastic two point boundary-value problems have received much less attention [1, 2]. In this paper we are interested in the numerical solutions of the following scalar stochastic two-point boundary value problems (STPBVP):

$$\frac{d^2u}{dt^2} + f(u, \frac{du}{dt}) = \frac{dW(t)}{dt}, \quad 0 \leqslant t \leqslant 1, \tag{1.1}$$

$$u(0) = a, \quad u(1) = b, \tag{1.2}$$

where $W = W(t),\ t > 0$ is an one-dimensional Brownian motion and $dW(t)/dt$ is the corresponding white noise, $f : \mathbb{R}^2 \to \mathbb{R}$ is a locally bounded function and a and b are constants. The existence and uniqueness of the solution of (1.1) was proved by Nualart and Pardoux [21, 22]. Numerical approximations of (1.1)-(1.2) were studied by several authors. In [1], Acriniega and

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Allen studied the shooting method for the corresponding linear Stratonovich stochastic boundary value problems. In [2], the shooting method was extended to system of nonlinear SDEs with boundary conditions.

One of the disadvantages of the shooting method is that its convergence can only be guaranteed if the initial point is in a sufficiently small neighborhood of the exact solution. Obviously this may cause some difficulty in practice since the exact solution is not known. In this paper, we propose to use the homotopy continuation method to find numerical solutions of (1.1)-(1.2). Similar to the shooting method, we will first transform the boundary value problem into an initial value problem with an unknown initial condition for the first derivative. Then we apply a homotopy continuation method to find the unknown initial value. We shall prove that under certain regularity conditions, the resulting numerical solution of the homotopy method converges at the same rate as the numerical algorithm used to solve the initial value problem. Our numerical experiments demonstrate that the homotopy continuation method may be less restrictive in selecting the initial iterative point than the shooting method. It should be noted that the convergence analysis of the shooting method studied in [1, 2] was incomplete and the method developed in this paper can be used to obtain the convergence rate of the shooting method.

The outline of the paper is as follows. In Section 2, we introduce the homotopy continuation method for (1.1)-(1.2) and demonstrate the solvability of the the homotopy equation. In Section 3, we construct numerical solutions of (1.1)-(1.2) using the homotopy continuation method and carry out the error analysis. Finally in Section 4 we conduct numerical experiments to verify our theoretical results and demonstrate the effectiveness of the proposed numerical method.

2. The Homotopy Continuation Method

2.1. The homotopy continuation method and its solvability

As illustrated in [1] the STPBVP (1.1)-(1.2) can be converted to solving the following nonlinear equation:

$$F(x) = u(1, x) - b = 0, \tag{2.1}$$

where $u = u(t, x)$ is the solution of following initial value problem

$$\frac{d^2u}{dt^2} + f(u, \frac{du}{dt}) = \frac{dW(t)}{dt}, \quad 0 \leqslant t \leqslant 1, \tag{2.2}$$

$$u(0) = a, \quad u'(0) = x. \tag{2.3}$$

Integrating (2.2) twice with respect to t, we obtain the corresponding integral equation of (2.2) as follows.

$$u(t, x) = a + tx - \int_0^t \int_0^s f(u(r, x), u'(r, x))drds + \int_0^t \int_0^s dW(r)drds, \quad 0 \le t \le 1. \tag{2.4}$$

To solve (2.1) with the homotopy continuation method, one first defines a homotopy function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $H(x, 1) = F(x)$ and $H(x, 0) = G(x)$, where $G : \mathbb{R} \to \mathbb{R}$ is a smooth map whose root is known. A typical convex homotopy function takes the form

$$H(x, \lambda) = \lambda F(x) + (1 - \lambda)G(x). \tag{2.5}$$