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SPLITTING SCHEMES FOR A NAVIER-STOKES-CAHN-HILLIARD MODEL FOR TWO FLUIDS WITH DIFFERENT DENSITIES*

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Abstract

In this work, we focus on designing efficient numerical schemes to approximate a thermodynamically consistent Navier-Stokes/Cahn-Hilliard problem given in [3] modeling the mixture of two incompressible fluids with different densities. The model is based on a diffuse-interface phase-field approach that is able to describe topological transitions like droplet coalescense or droplet break-up in a natural way. We present a splitting scheme, decoupling computations of the Navier-Stokes part from the Cahn-Hilliard one, which is unconditionally energy-stable up to the choice of the potential approximation. Some numerical experiments are carried out to validate the correctness and the accuracy of the scheme, and to study the sensitivity of the scheme with respect to different physical parameters.

Mathematics subject classification: 35Q35, 65M60, 76D05, 76D45, 76T10.
Key words: Two-phase flow, Diffuse-interface phase-field, Cahn-Hilliard, Navier-Stokes, Energy stability, Variable density, Mixed finite element, Splitting scheme.

1. Introduction

The evolution in time of the interface of two or more immiscible fluids arise naturally in hydrodynamics and materials science for modeling many current scientific, engineering, and industrial applications.

In recent times, the diffuse-interface approach has been used to describe the dynamic of the interfaces by layers of small thickness. One fundamental advantage of these models is that they are able to describe topological transitions like droplet coalescense or droplet break-up in a natural way.

The diffuse-interface theory was originally developed as methodology for modeling and approximating solid-liquid phase transitions. This idea can be traced to van der Waals [25], and is the foundation for the phase-field theory for phase transition and critical phenomena. Thus,

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the structure of the interface is determined by molecular forces; the tendencies for mixing and de-mixing are balanced through a non-local mixing energy.

Hohenberg and Halperin presented in [19] the so-called *Model H*, in order to model two incompressible, viscous Newtonian fluids with the same density. In [18], Gurtin et al. arrived at the same model by using the rational continuum mechanics framework and showed that it satisfies the second law of thermodynamics, leading in both cases to the so-called Navier-Stokes/Cahn-Hilliard system (NSCH).

There are many works devoted to study numerical schemes to approximate NSCH model that in the most of cases consists on coupling schemes previously presented for Navier-Stokes (see for instance [12]) and Cahn-Hilliard system [8–10]. In [22], the author study numerically a NSCH model (for the case of three phases) presenting a splitting and unconditionally stable scheme (satisfying a discrete energy law).

In the last years, many authors have been concerned in designing models to describe the flow of two incompressible, viscous Newtonian fluids with different densities. Lowengrub and Truskinovsky derived in [21] a thermodynamically consistent extension of the NSCH model with different densities but in this case the velocity field is no longer divergence free, leading to new difficulties to design fully discrete numerical schemes. Recently, Abels has discussed in [1] about the existence of local in time strong solutions of the system derived in [21]. A new related approach has been presented in [24] where mass and volume conservation is obtained for binary fluids and some splitting numerical schemes are proposed, although no discrete energy laws satisfied by the schemes are provided.

In [4], Boyer gives the complete derivation and a numerical approach of a model for the study of incompressible two-fluids mixture with different densities and viscosities, although no energy law of the system is presented.

There is also an increasing interest in more general models that are able to capture the mixture of different complex fluids. We refer the reader to [26] for a general formulation using the diffuse-interface method, to [7] for energy stable schemes for the Cahn-Hilliard-Brinkman equation and to [6] for energy stable schemes for anisotropic Cahn-Hilliard systems.

On the other hand, Shen and Yang presented in [23] a model and numerical approximations for two-phase incompressible flows with different densities and viscosities. In [27], Zhang and Wang present a study of the influence of the mobility term in a model of two-phase incompressible flows with different densities but no physical background of the derivation of the model is presented or cited. Finally, Abels et al. derived in [3] a new thermodynamically consistent model for incompressible two-phase flows with different densities while in [2] the existence of weak solutions for this model is proved. For a recent review in multi-component mixtures using phase field models we refer the reader to [20].

In this work, we present an unconditionally energy-stable and splitting scheme to approximate the model derived by Abels et al. in [3], showing its validity by several numerical simulations. In fact, the scheme decouples computations of the Navier-Stokes part from the Cahn-Hilliard one, and it is unconditionally energy-stable up to the choice of the potential approximation.

The rest of the paper is organized as follows. In Section 2, we detail the model considered and its energy law. The splitting numerical scheme and its energy-stability are derived in Section 3. In Section 4 we present some $2D$ and $3D$ numerical simulations and we state some conclusions in Section 5. Finally, the well-posedness of the scheme is proved in an Appendix.