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# HOMOTOPY CONTINUATION METHODS FOR STOCHASTIC TWO-POINT BOUNDARY VALUE PROBLEMS DRIVEN BY ADDITIVE NOISES*

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**Abstract**

In this paper the homotopy continuation method for stochastic two-point boundary value problems driven by additive noises is studied. The existence of the solution of the homotopy equation is proved. Numerical schemes are constructed and error estimates are obtained. Numerical experiments demonstrate the effectiveness of the homotopy continuation method over other commonly used methods such as the shooting method.

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*Key words:* Stochastic differential equations, Homotopy method, Shooting method.

## 1. Introduction

In recent years much progress has been made in numerical methods for initial value problems of stochastic differential equations and stochastic partial differential equations see, e.g., [10,13,19,23,25,27,28]. In contrast, numerical methods for stochastic two point boundary-value problems have received much less attention [1, 2]. In this paper we are interested in the numerical solutions of the following scalar stochastic two-point boundary value problems (STPBVP):

$$\frac{d^2u}{dt^2} + f(u, \frac{du}{dt}) = \frac{dW(t)}{dt}, \quad 0 \leqslant t \leqslant 1, \tag{1.1}$$

$$u(0) = a, \quad u(1) = b, \tag{1.2}$$

where $W = W(t),\ t > 0$ is an one-dimensional Brownian motion and $dW(t)/dt$ is the corresponding white noise, $f : \mathbb{R}^2 \to \mathbb{R}$ is a locally bounded function and $a$ and $b$ are constants. The existence and uniqueness of the solution of (1.1) was proved by Nualart and Pardoux [21, 22]. Numerical approximations of (1.1)-(1.2) were studied by several authors. In [1], Acriniega and

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Allen studied the shooting method for the corresponding linear Stratonovich stochastic boundary value problems. In [2], the shooting method was extended to system of nonlinear SDEs with boundary conditions.

One of the disadvantages of the shooting method is that its convergence can only be guaranteed if the initial point is in a sufficiently small neighborhood of the exact solution. Obviously this may cause some difficulty in practice since the exact solution is not known. In this paper, we propose to use the homotopy continuation method to find numerical solutions of (1.1)-(1.2). Similar to the shooting method, we will first transform the boundary value problem into an initial value problem with an unknown initial condition for the first derivative. Then we apply a homotopy continuation method to find the unknown initial value. We shall prove that under certain regularity conditions, the resulting numerical solution of the homotopy method converges at the same rate as the numerical algorithm used to solve the initial value problem. Our numerical experiments demonstrate that the homotopy continuation method may be less restrictive in selecting the initial iterative point than the shooting method. It should be noted that the convergence analysis of the shooting method studied in [1, 2] was incomplete and the method developed in this paper can be used to obtain the convergence rate of the shooting method.

The outline of the paper is as follows. In Section 2, we introduce the homotopy continuation method for (1.1)-(1.2) and demonstrate the solvability of the the homotopy equation. In Section 3, we construct numerical solutions of (1.1)-(1.2) using the homotopy continuation method and carry out the error analysis. Finally in Section 4 we conduct numerical experiments to verify our theoretical results and demonstrate the effectiveness of the proposed numerical method.

## 2. The Homotopy Continuation Method

### 2.1. The homotopy continuation method and its solvability

As illustrated in [1] the STPBVP (1.1)-(1.2) can be converted to solving the following nonlinear equation:

$$F(x) = u(1, x) - b = 0, \tag{2.1}$$

where $u = u(t, x)$ is the solution of following initial value problem

$$\frac{d^2u}{dt^2} + f(u, \frac{du}{dt}) = \frac{dW(t)}{dt}, \quad 0 \leqslant t \leqslant 1, \tag{2.2}$$

$$u(0) = a, \quad u'(0) = x. \tag{2.3}$$

Integrating (2.2) twice with respect to $t$, we obtain the corresponding integral equation of (2.2) as follows.

$$u(t, x) = a + tx - \int_0^t \int_0^s f(u(r, x), u'(r, x))drds + \int_0^t \int_0^s dW(r)drds, \quad 0 \leq t \leq 1. \tag{2.4}$$

To solve (2.1) with the homotopy continuation method, one first defines a homotopy function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $H(x, 1) = F(x)$ and $H(x, 0) = G(x)$, where $G : \mathbb{R} \to \mathbb{R}$ is a smooth map whose root is known. A typical convex homotopy function takes the form

$$H(x, \lambda) = \lambda F(x) + (1 - \lambda)G(x). \tag{2.5}$$

Two of the most commonly used homotopy methods are the Newton homotopy and the fixed point homotopy. For the Newton homotopy, one chooses $G(x) = F(x) - F(x_0)$, i.e.,

$$H(x,\lambda) = F(x) - (1-\lambda)F(x_0), \tag{2.6}$$

and the homotopy continuation method is to find $x = x(\lambda)$ such that

$$H(x(\lambda),\lambda) = F(x(\lambda)) - (1-\lambda)F(x_0) = 0, \quad \lambda \in [0,1], \tag{2.7}$$

where $x_0$ is a starting point. For the fixed-point homotopy, one chooses $G(x) = x - x_0$, i.e.,

$$H(x,\lambda) = \lambda F(x) + (1-\lambda)(x - x_0), \tag{2.8}$$

and the homotopy continuation method is to find $x = x(\lambda)$ such that

$$H(x(\lambda),\lambda) = \lambda F(x) + (1-\lambda)(x - x_0) = 0, \quad \lambda \in [0,1], \tag{2.9}$$

where $x_0$ is a starting point.

There has been a large amount of work on the deterministic homotopy continuation method, see, e.g., [8, 14–16, 20, 29–31]. Since $F$ is a function depending a random variable, solving (2.7) is essentially a stochastic homotopy problem. Of course the first concern of solving such a stochastic homotopy problem is the existence of a solution of the homotopy Eq. (2.7) or (2.9) and the regularity of the homotopy solution. The regularity of $x = x(\lambda)$ as a function of $\lambda$ is especially important for the homotopy continuation method since it affects the feasibility and efficiency of the method. This is in general a difficult problem when $f$ appeared in (1.1) is an arbitrary nonlinear function. In what follows we provide an existence and regularity result when $f$ belongs to a somewhat restrictive, but non-trivial function class.

**Proposition 2.1.** *Assume that $f$ in* (1.1) *is given by*

$$f(u', u) = \alpha u' + g(u)$$

*where $\alpha$ is a constant and $g$ is a smooth function such that $\|g'\|_\infty \le K$ for a positive constant $K$. Then $u = u(t,x)$ is Lipschitz continuous respect to $x$. Furthermore, assume that*

$$|\alpha| + K < \ln 2, \tag{2.10}$$

*then for $H$ defined by* (2.6) *or* (2.8) *, there exists a unique solution $x = x(\lambda)$ to $H(x,\lambda) = 0$ and the solution is Lipschitz continuous.*

*Proof.* Denote by $u_x(t,x)$ the partial derivative of $u$ with respect to $x$. Differentiating (2.4) with respect to $x$, we have that

$$u_x(t,x) = t - \int_0^t (\alpha u_x(s,x) + \int_0^s g'(u(r,x))u_x(r,x)dr)ds, \quad 0 \le t \le 1. \tag{2.11}$$

Thus

$$\begin{aligned} |u_x(t,x)| &\le t + \int_0^t (|\alpha| + (t-s)|g'(u(s,x))|)\, |u_x(s,x)|ds \\ &\le t + \int_0^t (|\alpha| + K)|u_x(s,x)|ds. \end{aligned}$$

By the Gronwall inequality we obtain

$$|u_x(t,x)| \le te^{(|\alpha|+K)t}, \tag{2.12}$$

which proves the Lipschitz continuous of $u$ with respect to $x$. To prove the existence of solution $x = x(\lambda)$ and its Lipschitz continuity, we substitute $u_x$ on the right hand side of (2.11) with (2.12) to obtain

$$u_x(t,x) \ge t - \int_0^t (|\alpha| + K)e^{(|\alpha|+K)s} ds = 1 + t - e^{(|\alpha|+K)t}.$$

This, together with (2.10), implies that

$$u_x(1,x) \ge 2 - e^{|\alpha|+K} > 0. \tag{2.13}$$

For the Newton homotopy we have that $\frac{\partial H}{\partial x} = u_x(1,x)$ and for the fixed point homotopy we have that

$$\frac{\partial H}{\partial x} = \lambda u_x(1,x) + 1 - \lambda.$$

Therefore, $\frac{\partial H}{\partial x} > 0$ for both cases because of (2.13). By the global implicit function theorem ([24]), both Eq. (2.7) and Eq. (2.9) have a unique solution $x = x(\lambda)$ for $0 < \lambda \le 1$ and the Lipschitz continuity of $x = x(\lambda)$ follows immediately from (2.13). □

### 2.2. Solving the homotopy equation as a nonlinear equation

Define a partition of $[0, 1]$:

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_M = 1.$$

Using an iterative approximation such as the Newton iteration, we obtain a sequence $\{x_i := x(\lambda_i)\}_{i=0}^{\infty}$ from

$$H(x, \lambda_i) = 0, \quad i = 1, \cdots, M, \tag{2.14}$$

where $x_0 = x(\lambda_0)$ is the initial point of the iteration. For the Newton iteration, we have

$$x_{n+1} = \begin{cases} x_n - \left[H'_x(x_n, \frac{n}{M})\right]^{-1} H(x_n, \frac{n}{M}), & n = 0, 1, \cdots, M-1, \\ x_n - \big(F'(x_n)\big)^{-1} F(x_n), & n = M, M+1, \cdots. \end{cases} \tag{2.15}$$

We may also use the quasi-Newton iteration to obtain $\{x_n\}$ as follows (see, e. g. Keller [11]).

$$x_{n+1} = \begin{cases} x_n - \left[H'_x(x_n, \frac{n}{M})\right]^{-1} H(x_n, \frac{n}{M}), & n = 0, 1, \cdots, M-1, \\ x_n - \big(F(x_n) - F(x_{n-1})\big)^{-1} F(x_n)(x_n - x_{n-1}), & n \ge M. \end{cases} \tag{2.16}$$

If $F(x)^2 - F(x)F''(x)$ is invertible, we can use the cubical Halley iteration (2.17):

$$x_{n+1} = \begin{cases} x_n - \left[H'_x(x_n, \frac{n}{M})\right]^{-1} H(x_n, \frac{n}{M}), \ n = 0, 1, \cdots, M-1, \\ x_n - \big(F'(x_n)^2 - F(x_n)F''(x_n)\big)^{-1} \big(2F(x_n)F'(x_n)\big), \ n \ge M. \end{cases} \tag{2.17}$$

More details on this iterative procedure can be found in [26].

## 3. Numerical Solution and Error Analysis

To solve (2.7) or (2.9) numerically, we need to to solve the initial value problem (2.2)-(2.3) using a numerical method. First we consider the discretization of (2.2)-(2.3) by Euler-Maruyama scheme. To this end we rewrite (2.2)-(2.3) as a first order system as follows. Let $U=(u,v)^T$ and $G(u,v)=(f(u,v),v)^T$. Then (2.2)-(2.3) is equivalent to the first order system

$$\begin{cases} dU = G(U)dt + (dW(t),0)^T, & 0 < t \le 1, \\ U(0) = (a,x)^T. \end{cases} \tag{3.1}$$

Given a uniform partition $0=t_0<t_1<\cdots<t_N=1$ with $h=t_1-t_0$, the one step Euler–Maruyama method of approximating (3.1) is to find $U_n^h=U_n^h(x)=(u_n^h(x),v_n^h(x))^T$ such that

$$\begin{cases} U_{n+1}^h = U_n^h + hG(U_n^h) + (\Delta W_n^h,0)^T, & n=0,1,\cdots N-1, \\ U_0^h = (a,x)^T, \end{cases} \tag{3.2}$$

where $\Delta W_n^h = W(t_n+h)-W(t_n)$. Let $F^h(x)=u_N^h(x)-b$. We definite the discrete Newton homotopy function $H^h=H^h(\lambda,x)$ as

$$H^h(\lambda,x) = F^h(x) - (1-\lambda)F^h(x_0), \quad 0\le\lambda\le 1. \tag{3.3}$$

Denote by $x^h(\lambda)$ the root of $H^h$, i.e.,

$$F^h(x^h(\lambda)) - (1-\lambda)F^h(x_0) = 0. \tag{3.4}$$

Similarly the discrete fixed point homotopy function $H^h=H^h(\lambda,x)$ is given by

$$H^h(\lambda,x) = \lambda F^h(x) + (1-\lambda)(x-x_0), \quad 0\le\lambda\le 1, \tag{3.5}$$

and the discrete fixed point homotopy solution $x^h(\lambda)$ is given by

$$\lambda F^h(x^h(\lambda)) + (1-\lambda)(x^h(\lambda)-x_0) = 0. \tag{3.6}$$

Following a proof similar to that of Proposition 2.1, we can show that under the condition (2.10), the discrete homotopy Eq. (3.3) has an unique solution $x^h=x^h(\lambda)$. Furthermore there exists a constant $C_1$ independent of $h$ and $\lambda$ such that

$$E(|x^h(\lambda)|^2) \le C_1. \tag{3.7}$$

**Theorem 3.1.** *Assume that the conditions of Proposition 2.1 hold, and $x(\lambda)$ and $x_h(\lambda)$ are the solutions of the Newton homotopy Eq. (2.7) and the discrete Newton homotopy Eq. (3.4), respectively. Then there exists a constant $C$ independent of $h$ and $\lambda$ such that*

$$E(|x(\lambda)-x^h(\lambda)|^2) \le Ch^2, \quad 0<\lambda\le 1, \tag{3.8}$$

$$E\left(\max_{1\le n\le N} |(u(t_n)-u_n^h(x^h(1))|^2\right) \le Ch^2. \tag{3.9}$$

*Proof.* From (2.7) and (3.4) we have that

$$\begin{aligned} F\big(x(\lambda)\big) - F^h\big(x^h(\lambda)\big) &= (1-\lambda)\big(F(x_0)-F^h(x_0)\big) \\ &= (1-\lambda)\big(u(1,x_0)-u_N^h(x_0)\big). \end{aligned}$$

Therefore

$$\begin{aligned} F\big(x(\lambda)\big) - F\big(x^h(\lambda)\big) &= F^h\big(x^h(\lambda)\big) - F\big(x^h(\lambda)\big) + F\big(x(\lambda)\big) - F^h\big(x^h(\lambda)\big) \\ &= -\Big(u\big(1, x^h(\lambda)\big) - u_N^h\big(x^h(\lambda)\big)\Big) + (1-\lambda)\big(u(1,x_0) - u_N^h(x_0)\big). \end{aligned} \tag{3.10}$$

By the standard error estimate for the one step Euler approximation for additive noise (see pp. 342–344 and p. 346 of [13]), there exists a constant $C_2$ such that

$$E\big(u(1,z) - u_N^h(z)\big)^2 \le C_2\big(1 + E(z^2)\big)h^2. \tag{3.11}$$

Thus from (3.7) and (3.10) we have that

$$E\Big(F\big(x(\lambda)\big) - F\big(x^h(\lambda)\big)\Big)^2 \le 2C_2(2 + C_1 + x_0^2)h^2.$$

By the proof of Proposition 2.1 we obtain

$$E\Big(\big(x(\lambda)\big) - x^h(\lambda)\Big)^2 \le \frac{2C_2(2 + C_1 + x_0^2)}{(2 - e^{|\alpha|+K})^2}h^2$$

which proves (3.9). The proof is complete. □

Similarly, if we replace the Newton homotopy with the fixed point homotopy (3.5) and (3.5), we have similar error estimates.

**Theorem 3.2.** *Assume that the conditions of Proposition* 2.1 *hold, and* $x(\lambda)$ *and* $x_h(\lambda)$ *are the solutions of the Newton homotopy Eq.* (2.9) *and the discrete fixed point homotopy Eq.* (3.5), *respectively. Then there exists a constant* $C$ *independent of* $h$ *and* $\lambda$ *such that*

$$E(|x(\lambda) - x^h(\lambda)|^2) \le Ch^2, \quad 0 < \lambda \le 1, \tag{3.12}$$

$$E\left(\max_{1\le n\le N} |(u(t_n) - u_n^h(x^h(1))|^2\right) \le Ch^2. \tag{3.13}$$

*Proof.* From (2.9) and (3.5) we have that

$$\lambda\Big(F\big(x(\lambda)\big) - F\big(x^h(\lambda)\big)\Big) + (1-\lambda)\big(x(\lambda) - x_h(\lambda)\big) = -\lambda\big(F(x_h) - F^h(x_h)\big).$$

By the definition of $F$ and $F^h$ and the mean value theorem, there exists $\xi$ between $x(\lambda)$ and $x^h(\lambda)$ such that

$$\big(\lambda u_x(1,\xi) + (1-\lambda)\big)\big(x(\lambda) - x_h(\lambda)\big) = -\lambda\Big(u(1, x^h(\lambda)) - u_N^h\big(x^h(\lambda)\big)\Big). \tag{3.14}$$

Notice that $\lambda u_x(1,\xi) + (1-\lambda) = 1$ for $\lambda = 0$. Also, from (2.13) we have that $\lambda u_x(1,\xi) + (1-\lambda) = u_x(1,\xi) \ge 2 - e^{|\alpha|+K}$ for $\lambda = 1$. Thus from (3.7), (3.14), and (3.11) we have that

$$E\Big(\big(x(\lambda)\big) - x^h(\lambda)\Big)^2 \le \frac{\lambda^2 C_1}{(2 - e^{|\alpha|+K})^2}h^2, \quad 0 < \lambda \le 1,$$

which proves (3.12). The proof of (3.13) is the same as that of Theorem 3.1. □

With (3.9) and (3.13), we have established the overall error estimate of the Newton homotopy method and the fixed point homotopy method. From the proof of Theorem 3.1 and Theorem 3.2 we can see that the overall convergence rate of the homotopy approximation is the same as the convergence rate of the numerical algorithm of solving the initial value problem (3.1). In particular, if we consider the following 1.5 order scheme (see (4.5), page 353 of [13])

$$\left\{\begin{array}{rl} u_{n+1}^h = & u_n^h + v_n^h h + \frac{1}{2} f(u_n^h, v_n^h) h^2 \\ v_{n+1}^h = & v_n^h + f(u_n^h, v_n^h) + f_u(u_n^h, v_n^h)\Delta z \\ & +\frac{1}{2}\left(v_n^h f_u(u_n^h, v_n^h) + f(u_n^h, v_n^h) f_v(u_n^h, v_n^h) + \frac{1}{2} f_{uu}(u_n^h, v_n^h)\right) h^2, \end{array}\right. \tag{3.15}$$

where

$$\Delta z = \frac{1}{2} h^{3/2}\left(U_1 + \frac{1}{\sqrt{3}} U_2\right)$$

and $U_1$ and $U_2$ are two independent $N(0,1)$ distributed random variables, then we have the following error estimate.

**Theorem 3.3.** *Assume that the conditions of Proposition* 2.1 *hold, the* 1.5 *order scheme* (3.15) *is used in replacement of Euler-Maruyama scheme* (3.2), *and either Newton homotopy or the fixed point homotopy is used for the numerical approximation. Then there exists a constant $C$ independent of $h$ and $\lambda$ such that*

$$E(|x(\lambda) - x^h(\lambda)|^2) \leq Ch^3, \qquad 0 < \lambda \leq 1,$$
$$E\left(\max_{1\leq n\leq N} |(u(t_n) - u_n^h(x^h(1))|^2\right) \leq Ch^3,$$

*where $x(\lambda)$ is given by* (2.7) *for the Newton homotopy and by* (2.9) *for the fixed point homotopy.* □

To find the solution $x^h(\lambda)$ from (3.4), we need to use an iterative scheme such as the ones outlined in (2.15), (2.16) and (2.17) with $H$ replaced by $H_h$ and $F$ replaced by $F_h$. For example, the iterative scheme (2.15) now becomes

$$x_{n+1}^h = \left\{\begin{array}{ll} x_n^h - [(H_h)'_x(x_n^h, \frac{n}{M})]^{-1} H_h(x_n^h, \frac{n}{M}), & n = 0, 1, \cdots, M-1, \\ x_n^h - (F'_h(x_n))^{-1} F_h(x_n), & n = M, M+1, \cdots. \end{array}\right. \tag{3.16}$$

With a SDE solver such as (3.2) and a nonlinear equation solver such as (3.16), the homotopy continuation algorithm now consists of the following four steps:

**Algorithm 3.1.**

**Step 1**: Input $N, M, x_0, \delta$, and set $n = 0$, $\Delta t := \frac{1}{N}$, $\lambda_i := \frac{i}{M}, i = 1, \cdots, M-1$.

**Step 2**: For $n = 0, 1, 2, \cdots$, use (3.16) to compute $x_{n+1}$.

**Step 3**: If $E(|x_{n+1} - x_n|) \geqslant \delta$, then go to step 2; otherwise go to Step 4.

**Step 4**: Let $x = x_N^h$ and compute the approximate solution $U_n^h$ using (3.2).

## 4. Numerical Results

In this section we conduct several numerical experiments to demonstrate our proposed homotopy continuation algorithm. In what follows, we use the Monte Carlo method to evaluate the expectation of a random variable $z$:

$$Mc(z) = \frac{1}{K}\sum_{i=1}^{K} z_i,$$

Table 4.1: Comparison between the shooting method and the homotopy continuation methods with initial point $x_0 = 1.0$.

| Step | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Shooting | $-0.6241$ | 0.1607 | $-0.0022$ | 4.7019e-009 | 0.0000 |
| $N$-Homotopy | 1.0000 | $-0.2746$ | 0.0746 | $-0.0020$ | 0.0000 |
| $F$-Homotopy | 1.0000 | $-0.4034$ | 0.1003 | $-0.0047$ | 0.0000 |

Table 4.2: Comparison between the shooting method and the homotopy continuation methods with initial point $x_0 = 1.4$.

| Step | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Shooting | $-1.5366$ | 1.9541 | $-3.5756$ | 14.9567 |
| $N$-Homotopy | 1.4000 | $-0.6964$ | 0.6800 | $-1.3171$ |
| $F$-Homotopy | 1.4000 | $-0.7842$ | 0.7922 | $-0.9741$ |
| Step | 5 | 6 | 7 | 8 |
| Shooting | $-326$ | 1.67e+5 | $-4.38$e+10 | 3.02e+21 |
| $N$-Homotopy | 1.3099 | $-1.2911$ | 1.2430 | $-1.1241$ |
| $F$-Homotopy | 1.2929 | $-1.3711$ | 1.5430 | $-1.4241$ |
| Step | 9 | 10 | 11 | 12 |
| Shooting | $-1.4$e+43 | 3.24e+86 | $-1.65$e+173 | Inf |
| $N$-Homotopy | 0.8589 | $-0.4079$ | 0.0431 | 0.0000 |
| $F$-Homotopy | 1.1594 | $-1.0723$ | 0.8133 | $-0.6050$ |
| Step | 13 | 14 | 15 | 16 |
| $F$-Homotopy | 0.4635 | $-0.2782$ | 0.0034 | 0.0000 |

where $z_i$ are random samples of $z$. In all the numerical experiments below, we choose $K = 10,000$.

**Example 1.** We first solve the stochastic two point boundary value problem:

$$\frac{d^2u}{dt^2} + f(u, \frac{du}{dt}) = \frac{dW(t)}{dt}, \quad 0 \leqslant t \leqslant 1, \tag{4.1}$$

$$u(0) = 0, \quad u(1) = 1, \tag{4.2}$$

where

$$f(u, v) = -100 \arctan(10u) + 100u - 100.$$

This problem satisfies the smoothness and monotonicity conditions on $f$ [21, 22]. Therefore, it has a unique solution. The corresponding initial value problem as a first order system is given by

$$\begin{aligned} &\frac{du}{dt} = v, \\ &\frac{dv}{dt} = 100 \arctan(10u) - 100u + 100 + \frac{dW(t)}{dt}, \\ &u(0) = 0, \quad v(0) = x. \end{aligned}$$

Table 4.3: Comparison between the shooting method and the homotopy continuation methods with initial point $x_0 = -3.3$.

| Step | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Shooting | 12 | −223 | 7.7e+4 | −9.4e+9 | 1.4e+20 |
| N-Homotopy | −3.3000 | 1.6786 | −0.9469 | 0.4447 | −0.1073 |
| F-Homotopy | −3.3000 | 1.8927 | −1.2721 | 0.8474 | −0.6226 |
| Step | 6 | 7 | 8 | 9 | 10 |
| Shooting | −3.1e+40 | 1.5e+81 | −3.7e+162 | Inf | - |
| N-Homotopy | 0.0057 | 0.0000 | - | - | - |
| F-Homotopy | 0.3884 | −0.2472 | 0.1172 | −0.0083 | 0.0000 |

Denote by $N$-Homotopy and $F$-Homotopy, respectively, the Newton homotopy method and fixed-point homotopy method. In Table 1 we list the values of $Mc(x_n)$ obtained from both the simple shooting method and our homotopy continuation methods using the Newton iteration with $M = N = 10$ and initial point $x_0 = 1.4$. It is clear from the table that both the shooting method and our homotopy continuation methods are convergent.

In Table 2 we list the results with the initial point chosen as $x_0 = -3.3$. It is clear from the table that our homotopy continuation methods are still convergent, however, the simple shooting method is divergent. Table 3 lists the results with initial point chosen as $x_0 = 1.0$. Once again the results indicate that our continuation methods are convergent while the simple shooting method is divergent.

**Example 2.** In this example, we verify the first order convergence of our Newton homotopy algorithm with Euler–Maruyama solver (3.2) and 1.5 order convergence with SDE solver (3.15).

Consider the second-order linear two-point stochastic boundary-value problem:

$$\frac{d^2u}{dt^2} + f(u, \frac{du}{dt}) = \frac{dW(t)}{dt}, \quad 0 \leqslant t \leqslant 1, \tag{4.3}$$

$$u(0) = 1, \quad u(1) = e^{-1/4}, \tag{4.4}$$

where

$$f(u, v) = -\frac{1}{8}u - \frac{1}{4}v.$$

Since $f$ satisfies condition (2.10), Proposition 1 guarantees the unique solvability of the homotopy equation. Furthermore, the error estimates (3.8) and (3.9) hold. The corresponding first order initial value problem is given by

$$\begin{aligned} \frac{du}{dt} &= v, \\ \frac{dv}{dt} &= \frac{1}{8}u + \frac{1}{4}v + \frac{dW(t)}{dt}, \\ u(0) &= 1, \quad v(0) = x. \end{aligned}$$

To verify the convergence orders, we arrange our simulations into $M$ batches of $K$ simulations in the following way. Denoting by $u_{i,j,n}$ the numerical approximation to $u_{i,j}(t_n)$ at step point $t_n$ in the $i \times j$-th simulation of all $K \times M$ simulations, we use the mean of absolute errors

$$\hat{\epsilon}(u) = \frac{1}{KM} \sum_{j=1}^{M} \sum_{i=1}^{K} \sqrt{\sum_{k=1}^{r} \left( u^k_{i,j,n} - u^k_{i,j}(t_n) \right)^2}$$

measure convergence order (see p. 312 of [13] for a detailed explanation). In our examples, we choose $r = 2$, $M = 20$ and $K = 250$.The reference solution is computed with the small timestep $h = 2^{-14}$. In Fig. 1, we plot the errors versus the time steps $h = 0.4 \times 2^{-i}$, $i = 1, \cdots, 8$ using the Euler-Maruyama solver. The reference line (broken) with slope 1.0 is plotted to verify the first order convergence. In Fig. 2, we plot the errors versus the time steps $h = 0.4 \times 2^{-i}$, $i = 1, \cdots, 8$ using the SDE solver of 1.5 order convergence (3.15) . The reference line (broken) with slope 1.5 is plotted to verify the 1.5 order convergence. Fig. 3 shows the graphs of Monte Carlo average $Mc(u_h)$ using the homotopy continuation method and a sample path of the numerical solution for $u$ for $h = 0.1$ and $h = 0.05$.

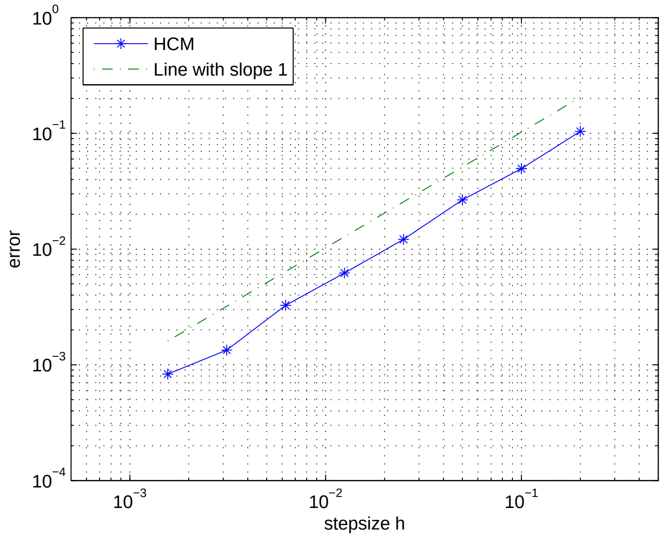


Fig. 4.1. Means of absolute errors for numerical solution of (4.3)-(4.4): Euler-Maruyama solver.

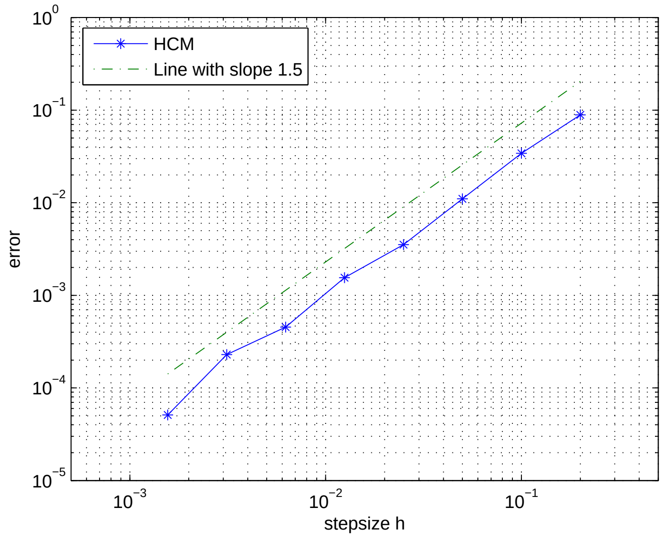


Fig. 4.2. Means of absolute errors for numerical solution of (4.3)-(4.4): 1.5 order solver.

**Example 3.** In the last example, we consider the nonlinear two point boundary value problem

$$\frac{d^2u}{dt^2} + f(u, \frac{du}{dt}) = \frac{dW(t)}{dt}, \quad 0 \leqslant t \leqslant 1, \tag{4.5}$$

$$u(0) = 0, \quad u(1) = \pi/6, \tag{4.6}$$

where

$$f(u, v) = -\frac{1}{2}\sin(u)\cos^{-3}(u).$$

This problem also satisfies the smoothness and monotonicity conditions on $f$ [21,22]. Therefore, it has a unique solution. The corresponding first order initial value problem is given by

$$\begin{aligned} &\frac{du}{dt} = v, \\ &\frac{dv}{dt} = \frac{1}{2}\sin(u)\cos^{-3}(u) + \frac{dW(t)}{dt}, \\ &u(0) = 1, \quad v(0) = x. \end{aligned}$$

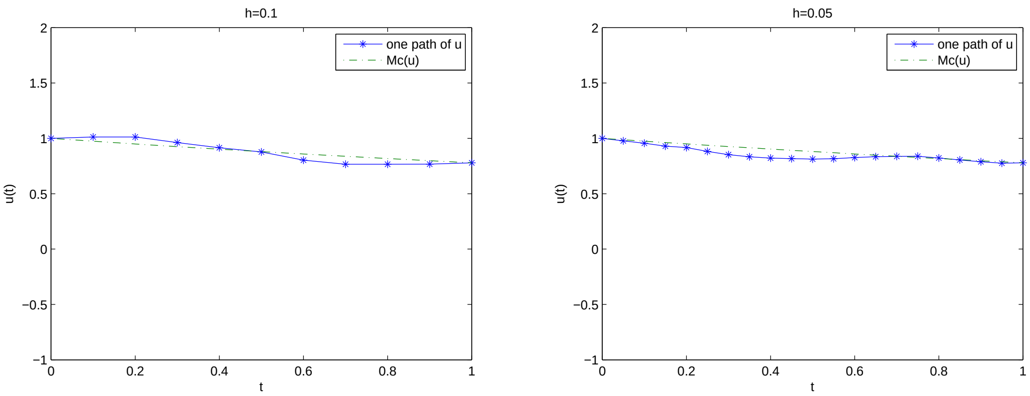


Fig. 4.3. Expectations and sample paths for solutions of (4.3)-(4.4).

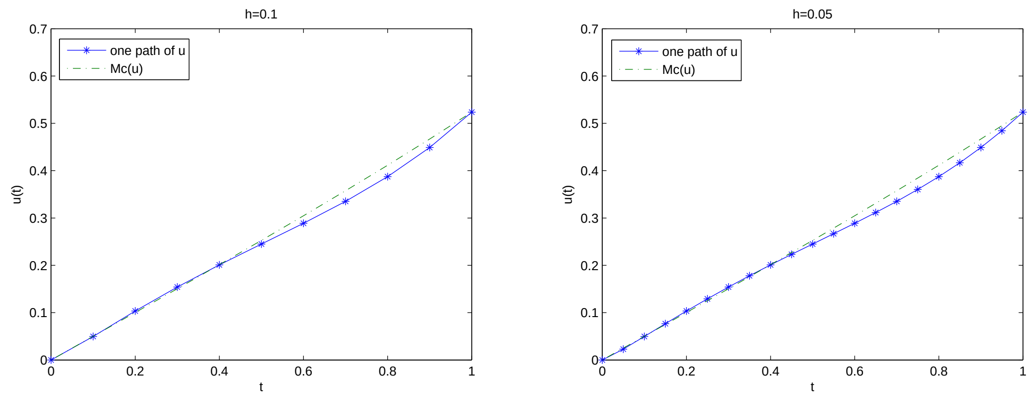


Fig. 4.4. Expectations and sample paths (4.5)-(4.6) using the homotopy continuation method procedure.

Graphs of approximate expectations using the Monte Carlo simulation and sample paths similar to Fig. 3 are shown in Fig. 4.

## 5. Conclusions

In this paper, we extended the deterministic homotopy continuation method to stochastic two point boundary value problems driven by additive white noises. With this method, the stochastic boundary value problem is solved by iteratively solving a sequence of initial value problems. We prove the solvability of the homotopy continuation equation under some regularity conditions for the nonlinear lower order terms and obtain error estimates. Our numerical results indicate that the proposed homotopy continuation method is less restrictive in choosing initial iterative point than the shooting method.

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