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UNIFORMLY CONVERGENT NONCONFORMING ELEMENT FOR 3-D FOURTH ORDER ELLIPTIC SINGULAR PERTURBATION PROBLEM*

Hongru Chen
College of Science, Henan University of Technology, Zhengzhou 450001, China
Email: chenhongru5@126.com
Shaochun Chen*
Department of Mathematics, Zhengzhou University, Zhengzhou 450001, China
Email: shchchen@zzu.edu.cn

Abstract

In this paper, using a bubble function, we construct a cuboid element to solve the fourth order elliptic singular perturbation problem in three dimensions. We prove that the nonconforming C^0-cuboid element converges in the energy norm uniformly with respect to the perturbation parameter.

Mathematics subject classification: 65N12, 65N30.
Key words: Nonconforming finite element, Singular perturbation problem, Uniform error estimates.

1. Introduction

Let $\Omega \subset R^3$ be a bounded polyhedral domain with boundary $\partial\Omega$. For $f \in L^2(\Omega)$, we consider finite element methods for the following boundary value problem of fourth order elliptic singular perturbation equation:

$$\begin{cases} \varepsilon^2\Delta^2 u - \Delta u = f, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Δ is the standard Laplace operator, $\partial u/\partial n$ denotes the outer normal derivative on $\partial\Omega$ and ε is a small real parameter with $0 < \varepsilon \le 1$. This problem can be considered a gross simplification of the stationary Cahn-Hilliard equation with ε being the length of the transition region of phase separation. In particular, we are interested in the regime when ε tends to zero. Obviously, if ε approaches zero, the differential Eq. (1.1) formally degenerates to Poisson's equation.

For ε=1, that is, the usual fourth order elliptic equation, many works have been done. When a conforming finite element is used, it should consist of piecewise polynomials that are globally continuously differentiable (C^1). Such elements require polynomials of high degree and even in two dimensions are not easy to construct. To lower the polynomial degree, some macroelements were created on triangle grids, see e.g., [1, 2]. Recently, a macro type of biquadratic C^1 finite element was constructed on rectangle grids [3, 4], which is a rectangular version of the (C^1) Powell-Sabin element [1]. On the other hand, many lower degree nonconforming elements in the two and three dimensional cases have been constructed and used in practice.

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For the fourth order elliptic singular perturbation problem (1.1), the Morley element is a nature choice for the biharmonic operator since it has the least number of degrees of freedom on each element for fourth order problems [5]. Unfortunately, this element is divergent for general second order elliptic problems [2,6-8], so we can not get the uniformly convergent result as $\varepsilon \to 0$ as was shown in [6]. As a result, in order to obtain robust schemes, either the formulation of the Morley element method must be modified or the element itself must be changed, and several variants of the Morley element method have been presented [6, 9, 10].

In the two-dimensional case, a nonconforming C^0 triangular element was constructed in [6], by enriching second degree polynomials with cubic bubble function. A modified triangular Morley element and a modified rectangular Morley element were presented in [9]. In that paper, the authors used the original Morley element and changed the discrete variational problem. An C^0 rectangular element was constructed in [10]. A nine parameter non-C^0 triangular element and a twelve parameter non-C^0 rectangular element were proposed in [11] by the double set parameter technique. Later, by the same technique, a nine parameter C^0 triangular element was analyzed in [12] and a non-C^0 rectangular element was constructed in [13], but the later paper was solving problem (1.1) but with boundary conditions $u = \partial^2 u/\partial n^2 = 0$. All of these nonconforming elements were proved to be uniformly convergent.

In the three-dimensional case. A nonconforming non-C^0 tetrahedral element was constructed and analyzed in [14] by the similar way used in [9], and a nonconforming C^0 tetrahedral element was constructed in [15]. Recently, a nonconforming C^0 tetrahedral element was constructed in [16] by Nitsche's method. In this paper, we introduce an C^0 cuboid element, which was constructed in [17] by us, but the error estimate was valid only for $\varepsilon = 1$. Here, we prove that the element is robust with respect to the perturbation parameter and uniforming convergent to order $h^{1/2}$. Moreover, besides the theoretical interest, our new finite element method is expected to be useful in the computation of the Cahn-Hilliard equation.

The rest of this paper is organized as follows. In the following section, we list some notations and two basic preliminaries. Next, we give detailed descriptions of the cuboid element. Finally, we prove the element is uniformly convergent in ε for the fourth order elliptic singular perturbation equation.

2. Preliminaries

For a nonnegative integer m, we shall use the standard notation $H^m(\Omega)$ to denote the Sobolev space of functions with partial derivatives up to m in $L^2(\Omega)$. The corresponding norm and semi-norm are denoted by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. The space $H_0^m(\Omega)$ is the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{m,\Omega}$ and $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$. P_k is the polynomial space of degree not greater than k and Q_k is the polynomial space of degree in each coordinate not greater than k.

Define

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{3} \partial_{ij} u \partial_{ij} v dx, \quad \forall u, v \in H^2(\Omega), \tag{2.1}$$

$$b(u,v) = \int_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v dx, \qquad \forall u, v \in H^1(\Omega). \tag{2.2}$$

The weak form of (1.1) is: find $u \in H_0^2(\Omega)$ such that

$$\varepsilon^2 a(u,v) + b(u,v) = f(v), \quad \forall v \in H_0^2(\Omega). \tag{2.3}$$

The energy norm of (2.3) is defined by

$$|||v|||_{\varepsilon}^2 = \varepsilon^2 a(v,v) + b(v,v) = \varepsilon^2 |v|_{2,\Omega}^2 + |v|_{1,\Omega}^2.$$

Let $\mathcal{T}_h$ be a division of Ω into cuboids with mesh size h, and $\{\mathcal{T}_h\}$ be a family of divisions with $h \to 0$. Throughout this paper, we assume that $\{\mathcal{T}_h\}$ is regular and quasi-uniform, namely, it satisfies that :

$$h_T/\rho_T \le \sigma_1, \;\; h_T/h_{T'} \le \sigma_2, \quad \forall T, T^{'} \in \mathcal{T}_h, \; \forall h,$$

where h_T and ρ_T are the diameters of T and the largest ball contained in T, respectively, $\sigma_1 > 0$, $\sigma_2 > 0$ are constants independent of h. The nonconforming finite element space V_h is a piecewise polynomial space such that $V_h \not\subset H_0^2(\Omega)$. The discrete problem of (2.3) is: find $u_h \in V_h$ satisfying

$$\varepsilon^2 a_h(u_h, v_h) + b_h(u_h, v_h) = f(v_h), \quad \forall v_h \in V_h, \tag{2.4}$$

where

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{i,j=1}^{3} \partial_{ij} u_h \partial_{ij} v_h dx, \tag{2.5}$$

$$b_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{i=1}^{3} \partial_i u_h \partial_i v_h dx. \tag{2.6}$$

The discrete energy norm is :

$$|||v_h|||_{\varepsilon,h}^2 = \varepsilon^2 a_h(u_h, v_h) + b_h(u_h, v_h) = \varepsilon^2 |v|_{2,h}^2 + |v|_{1,h}^2,$$

where $|\cdot|_{i,h}^2 = \sum\limits_{T \in \mathcal{T}_h} |\cdot|_{i,T}^2$, $i = 1, 2$.

The following result is well known as the Strang Lemma [2, 18].

Lemma 2.1. *Let u and u_h be the solutions of* (2.3) *and* (2.4)*, then*

$$|||u - u_h|||_{\varepsilon,h} \le C \left(\inf_{v_h \in V_h} |||u - v_h|||_{\varepsilon,h} + \sup_{w_h \in V_h} \frac{|a_h(u, w_h) + b_h(u, w_h) - f(w_h)|}{|||w_h|||_{\varepsilon,h}} \right), \tag{2.7}$$

where $C > 0$ is a constant independent of h.

The first term of (2.7) is the approximation error and the second term of (2.7) is the consistency error.

Let $F \subset \partial T$ be a face of T and $\mathcal{F}_h = \{F; F \subset \partial T, T \in \mathcal{T}_h\}$. Suppose $F = T \cap T^{'}$, define

$$[w]|_F = w|_{T \cap F} - w|_{T' \cap F}; \quad [w]|_F = w|_F, \quad \text{if } \; F \subset \partial\Omega.$$

For any $F \subset \partial T$, $\forall T \in \mathcal{T}_h$, let $n = (n_1, n_2, n_3)^\top$ be the unit outer normal vector to F, τ, s be two unit vectors and orthogonal to each other on F, and they constitute the right hand coordinate system. Then we have

$$\partial_j = n_j \partial_n + \tau_j \partial_\tau + s_j \partial_s, \quad 1 \le j \le 3,$$

where

$$\partial_j = \frac{\partial}{\partial x_j}, \; \partial_\tau = \frac{\partial}{\partial \tau}, \; \partial_s = \frac{\partial}{\partial s}, \; \partial_n = \frac{\partial}{\partial n}.$$

If $V_h \subset H_0^1(\Omega)$, then for all $F \subset \mathcal{F}_h$, $w_h \in V_h$, we have $[w_h]|_F = [\partial_\tau w_h]|_F = [\partial_s w_h]|_F = 0$. From Green formula [2, 18], we derive

$$\begin{aligned}
&\varepsilon^2 a_h(u, w_h) + b_h(u, w_h) \\
= &\varepsilon^2 \sum_{T\in\mathcal{T}_h} \int_T \sum_{i,j=1}^{3} \partial_{ij} u \partial_{ij} w_h dx + \sum_{T\in\mathcal{T}_h} \int_T \sum_{i=1}^{3} \partial_i u \partial_i w_h dx \\
= &\varepsilon^2 \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{3} \{\int_{\partial T} \partial_{ij} u \partial_j w_h n_i d\tau ds + \int_T \partial_{iijj} u w_h dx\} - \sum_{T\in\mathcal{T}_h} \int_T \Delta u w_h dx \\
= &\varepsilon^2 \sum_{T\in\mathcal{T}_h} \int_{\partial T} \sum_{i,j=1}^{3} \partial_{ij} u (n_j \partial_n w_h + \tau_j \partial_\tau w_h + s_j \partial_s w_h) n_i d\tau ds \\
&+ \varepsilon^2 \sum_{T\in\mathcal{T}_h} \int_T \Delta^2 u w_h dx - \sum_{T\in\mathcal{T}_h} \int_T \Delta u w_h dx \\
= &\varepsilon^2 \sum_{T\in\mathcal{T}_h} \int_{\partial T} \partial_{nn} u \partial_n w_h d\tau ds + \varepsilon^2 \sum_{T\in\mathcal{T}_h} \int_T \Delta^2 u w_h dx - \sum_{T\in\mathcal{T}_h} \int_T \Delta u w_h dx.
\end{aligned}$$

Since $\varepsilon^2 \Delta^2 u - \Delta u = f$, we have

$$\varepsilon^2 a_h(u, w_h) + b_h(u, w_h) - f(w_h) = \varepsilon^2 \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \int_F \partial_{nn} u \partial_n w_h d\tau ds, \quad \forall w_h \in V_h. \tag{2.8}$$

In [14], Wang and Meng derived the following refined regularity result:

Lemma 2.2. *If Ω is convex, then there exists a constant C independent of ε such that*

$$\varepsilon^{-\frac{1}{2}} |u - u^0|_{1,\Omega} + \varepsilon^{\frac{1}{2}} |u|_{2,\Omega} + \varepsilon^{\frac{3}{2}} |u|_{3,\Omega} \le C\|f\|_{0,\Omega}, \tag{2.9}$$

for all $f \in L^2(\Omega)$. where u^0 is the solution of following reduced problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

3. Nonconforming C^0-Cuboid Element

Let $\hat{T} = [-1, 1]^3$ be the reference element with nodes

$$\hat{a}_1(-1,-1,-1), \quad \hat{a}_2(1,-1,-1), \quad \hat{a}_3(1,1,-1), \quad \hat{a}_4(-1,1,-1),$$
$$\hat{a}_5(-1,-1,1), \quad \hat{a}_6(1,-1,1), \quad \hat{a}_7(1,1,1), \quad \hat{a}_8(-1,1,1).$$

The 6 faces of $\hat{T}$ are defined by

$$\hat{F}_1 = \Box\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4, \quad \hat{F}_2 = \Box\hat{a}_5\hat{a}_6\hat{a}_7\hat{a}_8, \quad \hat{F}_3 = \Box\hat{a}_1\hat{a}_5\hat{a}_6\hat{a}_2,$$
$$\hat{F}_4 = \Box\hat{a}_4\hat{a}_8\hat{a}_7\hat{a}_3, \quad \hat{F}_5 = \Box\hat{a}_1\hat{a}_4\hat{a}_8\hat{a}_5, \quad \hat{F}_6 = \Box\hat{a}_2\hat{a}_3\hat{a}_7\hat{a}_6.$$

The 12 edges of $\hat{T}$ are defined by

$$\hat{l}_1 = \overline{\hat{a}_1\hat{a}_2}, \quad \hat{l}_2 = \overline{\hat{a}_3\hat{a}_4}, \quad \hat{l}_3 = \overline{\hat{a}_7\hat{a}_8}, \quad \hat{l}_4 = \overline{\hat{a}_5\hat{a}_6}, \quad \hat{l}_5 = \overline{\hat{a}_1\hat{a}_4}, \quad \hat{l}_6 = \overline{\hat{a}_2\hat{a}_3},$$

$$\hat{l}_7 = \overline{\hat{a}_6\hat{a}_7}, \quad \hat{l}_8 = \overline{\hat{a}_5\hat{a}_8}, \quad \hat{l}_9 = \overline{\hat{a}_1\hat{a}_5}, \quad \hat{l}_{10} = \overline{\hat{a}_2\hat{a}_6}, \quad \hat{l}_{11} = \overline{\hat{a}_3\hat{a}_7}, \quad \hat{l}_{12} = \overline{\hat{a}_4\hat{a}_8}.$$

The middle points of $\hat{l}_i$ is denoted by $\hat{g}_i$ $(1 \le i \le 12)$. See Fig. 3.1. Let

$$b_{\hat{T}} = (1-\hat{x}_1^2)(1-\hat{x}_2^2)(1-\hat{x}_3^2).$$

Then $b_{\hat{T}}$ is the bubble function satisfying $b_{\hat{T}} \in Q_2(\hat{T}),\ b_{\hat{T}}|_{\hat{F}_i} = 0,\ 1 \le i \le 6.$

The shape function space for C^0 cuboid element is taken as:

$$\hat{P} = \hat{P}_2^* \oplus b_{\hat{T}}\{\,\hat{x}_i,\ \hat{x}_i^2,\ 1 \le i \le 3\}, \tag{3.1}$$

where $\hat{P}_2^* = P_2(\hat{T}) \oplus \{\,\hat{x}_1\hat{x}_2\hat{x}_3,\ \hat{x}_i^2\hat{x}_{i+1},\ \hat{x}_i^2\hat{x}_{i+2},\ \hat{x}_i^2\hat{x}_{i+1}\hat{x}_{i+2},\ 1 \le i \le 3,\ mod\,3\}$. Hence, the space $\hat{P}$ is a linear space of dimension 26. The degrees of freedom are given as follows:

$$\Sigma_{\hat{T}} = \left\{ \hat{v}(\hat{a}_i),\ 1 \le i \le 8, \quad \hat{v}(\hat{g}_i),\ 1 \le i \le 12, \quad \int_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} d\hat{s},\ 1 \le i \le 6 \right\}. \tag{3.2}$$

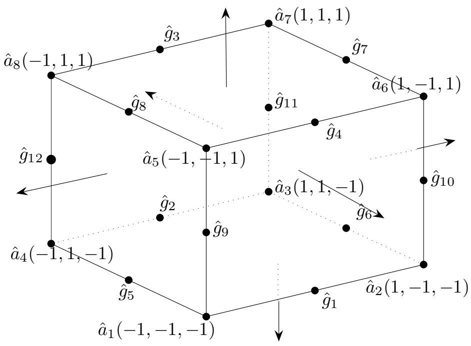

Fig. 3.1. Degrees of freedom of C^0 cuboid element.

Lemma 3.1. ([17]) *Any function $w \in \hat{P}$ is uniquely determined by the degrees of freedom* (3.2), *namely, $\Sigma_{\hat{T}}$ is $\hat{P}$-unisolvent.*

For cuboid mesh $\mathcal{T}_h$, let $T \in \mathcal{T}_h$ be an element with center (x_{10}, x_{20}, x_{30}) and $2h_{T1}$, $2h_{T2}$, $2h_{T3}$ be the lengths of T along x_1, x_2, x_3 coordinates, respectively. The affine transformation $x = F(\hat{x}) : \hat{T} \to T$ is

$$x_i = h_{Ti}\hat{x}_i + x_{i0}, \quad 1 \le i \le 3.$$

Under $x = F(\hat{x})$, let $\hat{a}_i \leftrightarrow a_i$, $1 \le i \le 8$; $\hat{F}_i \leftrightarrow F_i$, $1 \le i \le 6$; $\hat{l}_i \leftrightarrow l_i$, $\hat{g}_i \leftrightarrow g_i$, $1 \le i \le 12$; $\hat{P} \leftrightarrow P_T$; $\hat{v}(\hat{x}) = v(x)$. Then the degrees of freedom of P_T on T are

$$v_i,\ 1 \le i \le 8, \quad v(g_i),\ 1 \le i \le 12, \quad \int_{F_i} \frac{\partial v}{\partial n} d\tau ds,\ 1 \le i \le 6. \tag{3.3}$$

The degrees of freedom (3.3) defines a local interpolation operator $\Pi_T : H^3(T) \to P_T$. It is easy to prove that the interpolation operate Π_T is affine interpolation equivalent.

The finite element space for the C^0 cuboid element is defined by

$$V_{h0} = \left\{ v_h : v_h|_T \in P_T,\ [v_h]|_F = 0,\ \int_F \left[\frac{\partial v_h}{\partial n}\right] d\tau ds = 0, \quad \forall F \subset \partial T,\ \forall T \in \mathcal{T}_h \right\}. \tag{3.4}$$

The corresponding finite element interpolation operator $\Pi_h : H^3(\Omega) \cap H_0^2(\Omega) \to V_{h0}$ is defined by $\Pi_h|_T = \Pi_T$, for all $T \in \mathcal{T}_h$. It is easy to prove that

$$V_{h0} \subset H_0^1(\Omega), \tag{3.5}$$

Then, the discrete variational problem using C^0 cuboid element to solve (2.3) is: find $u_h \in V_{h0}$ such that

$$\varepsilon^2 a_h(u_h, v_h) + b(u_h, v_h) = f(v_h), \quad \forall v_h \in V_{h0}. \tag{3.6}$$

4. Convergence Analysis

In this section, we discuss the convergence properties of the C^0 cuboid element given in the previous section.

It is easy to check that $|||\cdot|||_{\varepsilon,h}$ is a norm of V_{h0}, so (3.6) are unisolvent by the Lax-Milgram Theorem [2, 18].

Because $P_2(T) \subset P_T$, namely, interpolation operator Π_h preserves quadratics locally, it follows from a standard scaling argument, using the Bramble-Hilbert lemma, that there exists a constant C independent of h such that

$$\sum_{T\in\mathcal{T}_h} \|v - \Pi_k v\|_{j,T} \le Ch^{k-j}|v|_{k,\Omega},\ j = 0, 1, 2;\ k = 2, 3, \quad \forall v \in H^k(\Omega). \tag{4.1}$$

Moreover, if $\hat{T}$ is a reference element, by using a Bramble-Hilbert argument and following the standard trace inequality [2, 18]

$$\|\hat{v}\|_{0,\partial\hat{T}} \le C\|\hat{v}\|_{0,\hat{T}}^{\frac{1}{2}}\|\hat{v}\|_{1,\hat{T}}^{\frac{1}{2}}, \tag{4.2}$$

we get

$$|v - \Pi v|_{1,\Omega} \le Ch^{\frac{1}{2}}|v|_{1,\Omega}^{\frac{1}{2}}|v|_{2,\Omega}^{\frac{1}{2}}, \quad \forall v \in H_0^2(\Omega). \tag{4.3}$$

By the definition of the space V_{h0}, we obtain the following convergence theorem for the C^0 cuboid element.

Theorem 4.1. *Assume that u is the weak solution of* (1.1) *for a given $f \in L^2(\Omega)$. Furthermore, let u_h be the discrete solutions of* (3.6)*. Then there exists a constant C, independent of ε and h, such that*

$$|||u - u_h|||_{\varepsilon,h} \le C \begin{cases} (h^2 + \varepsilon h)|u|_{3,\Omega}, \\ h(|u|_{2,\Omega} + \varepsilon|u|_{3,\Omega}). \end{cases}$$

Proof. By Lemma 2.1

$$|||u - u_h|||_{\varepsilon,h} \le C\left(\inf_{v_h\in V_{h0}} |||u - v_h|||_{\varepsilon,h} + \sup_{w_h\in V_h} \frac{|E_{\varepsilon,h}(u, w_h)|}{|||w_h|||_{\varepsilon,h}}\right). \tag{4.4}$$

The interpolation estimate (4.1) implies that

$$\inf_{v_h\in V_h} |||u - v_h|||_{\varepsilon,h} \le |||u - \Pi_h u|||_{\varepsilon,h} \le C \begin{cases} (h^2 + \varepsilon h)|u|_{3,\Omega}, \\ h(|u|_{2,\Omega} + \varepsilon|u|_{3,\Omega}). \end{cases} \tag{4.5}$$

Hence, it remains to estimate $E_{\varepsilon,h}(u, w_h)$. Put

$$P_F v = \frac{1}{|F|}\int_F v d\tau ds,\ P_T v = \frac{1}{|T|}\int_T v d\tau dx.$$

Then

$$P_F v = \frac{1}{|F|}\int_F v d\tau ds = \frac{1}{|\hat{F}|}\int_{\hat{F}} \hat{v} d\hat{\tau} d\hat{s} = P_{\hat{F}}\hat{v},$$
$$P_T v = \frac{1}{|T|}\int_T v dx = \frac{1}{|\hat{T}|}\int_{\hat{T}} \hat{v} d\hat{x} = P_{\hat{T}}\hat{v}.$$

Since $u \in H^3(\Omega)$, from (2.8) and the definition of the space V_{h0}, we have

$$\begin{aligned} E_{\varepsilon,h}(u, w_h) &= \varepsilon^2 a_h(u, w_h) + b_h(u, w_h) - f(w_h) \\ &= \varepsilon^2 \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F \partial_{nn}u\partial_n w_h d\tau ds, \\ &= \varepsilon^2 \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F (\partial_{nn}u - P_T\partial_{nn}u)(\partial_n w_h - P_F\partial_n w_h) d\tau ds. \end{aligned} \tag{4.6}$$

Set $\mu = \partial_{nn}u$ and $\varphi = \partial_n w_h$. Then from trace theorem [19] and scaling argument, we obtain

$$\begin{aligned} &\left|\int_F (\partial_{nn}u - P_T\partial_{nn}u)(\partial_n w_h - P_F\partial_n w_h) d\tau ds\right| \\ = &\left|\int_F (\mu - P_T\mu)(\varphi - P_F\varphi) d\tau ds\right| \\ \le &Ch^2\|\hat{\mu} - P_{\hat{T}}\hat{\mu}\|_{0,\hat{F}}\|\hat{\varphi} - P_{\hat{F}}\hat{\varphi}\|_{0,\hat{F}} \le Ch^2\|\hat{\mu} - P_{\hat{T}}\hat{\mu}\|_{1,\hat{T}}\|\hat{\varphi} - P_{\hat{F}}\hat{\varphi}\|_{1,\hat{T}} \\ \le &Ch^2|\hat{\mu}|_{1,\hat{T}}|\hat{\varphi}|_{1,\hat{T}} \le Ch|\mu|_{1,T}|\varphi|_{1,T} \le Ch|u|_{3,T}|w_h|_{2,T}. \end{aligned} \tag{4.7}$$

Putting (4.7) into (4.6) we get

$$\sup_{w_h\in V_{h0}} \frac{|E_{\varepsilon,h}(u, w_h)|}{|||w_h|||_{\varepsilon,h}} \le C\varepsilon h|u|_{3,\Omega}, \tag{4.8}$$

and together with (4.4) and (4.5) this implies the desired estimates. □

The regularity result of Lemma 2.2 given in the Section 2 leads to the following uniform convergence property for the nonconforming finite element method (3.6).

Theorem 4.2. *Suppose that u and u_h are the solutions of* (2.3) *and* (3.6)*, respectively, there exists a constant $C > 0$ independent of h, ε and f, such that*

$$|||u - u_h|||_{\varepsilon,h} \le Ch^{\frac{1}{2}}\|f\|_{0,\Omega}.$$

Proof. Similar to the proof of Theorem 4.1 we start with the basic estimate (4.4). Throughout this proof, we assume that C denotes a constant independent of ε, h and f. We first show that

$$\inf_{v_h\in V_h} |||u - v_h|||_{\varepsilon,h} \le |||u - \Pi_h u|||_{\varepsilon,h} \le Ch^{\frac{1}{2}}\|f\|_{0,\Omega}. \tag{4.9}$$

By (4.1) and Lemma 2.2, we get

$$\begin{aligned} &\varepsilon^2|u - \Pi_h u|^2_{2,h} \le c\varepsilon^2|u|_{2,h}|u - \Pi_h u|_{2,h} \\ \le &Ch\left(\varepsilon^{\frac{1}{2}}|u|_{2,h}\right)\left(\varepsilon^{\frac{3}{2}}|u|_{3,h}\right) \\ \le &Ch\|f\|^2_{0,\Omega}. \end{aligned} \tag{4.10}$$

Then, we have

$$\varepsilon|u-\Pi_h u|_{2,h}\le Ch^{\frac{1}{2}}\|f\|_{0,\Omega}.$$

In order to estimate the H^1-part of the energy norm we use the triangle inequality to obtain

$$|u-\Pi_h u|_{1,\Omega}\le |u-u^0-\Pi_h(u-u^0)|_{1,\Omega}+|u^0-\Pi_h u^0|_{1,\Omega}.$$

From (4.3) and Lemma 2.2 it follows that

$$\begin{aligned}&|u-u^0-\Pi_h(u-u^0)|_{1,\Omega}\le Ch^{\frac{1}{2}}|u-u^0|_{1,\Omega}^{\frac{1}{2}}|u-u^0|_{2,\Omega}^{\frac{1}{2}}\\ =&Ch^{\frac{1}{2}}\left(\varepsilon^{-\frac{1}{2}}|u-u^0|_{1,\Omega}\right)^{\frac{1}{2}}\left(\varepsilon^{\frac{1}{2}}|u-u^0|_{2,\Omega}\right)^{\frac{1}{2}}\le Ch^{\frac{1}{2}}\|f\|_{0,\Omega},\end{aligned}$$

while (4.1) gives

$$|u^0-\Pi_h u^0|_{1,\Omega}\le Ch\|u^0\|_{2,\Omega}\le Ch\|f\|_{0,\Omega}. \tag{4.11}$$

So, we get the approximation error (4.9).

Using (4.2), the estimate (4.7) can be replaced by

$$\begin{aligned}&\left|\int_F(\partial_{nn}u-P_T\partial_{nn}u)(\partial_n w_h-P_F\partial_n w_h)d\tau ds\right|\\ =&\left|\int_F(\mu-P_T\mu)(\varphi-P_F\varphi)d\tau ds\right|\le Ch^2\|\hat\mu-P_{\hat T}\hat\mu\|_{0,\hat F}\|\hat\varphi-P_{\hat F}\hat\varphi\|_{0,\hat F}\\ \le& Ch^2\|\hat\mu-P_{\hat T}\hat\mu\|_{0,\hat T}^{\frac{1}{2}}\|\hat\mu-P_{\hat T}\hat\mu\|_{1,\hat T}^{\frac{1}{2}}\|\hat\varphi-P_{\hat F}\hat\varphi\|_{1,\hat T}\\ \le& Ch^2\|\hat\mu\|_{0,\hat T}^{\frac{1}{2}}|\hat\mu|_{1,\hat T}^{\frac{1}{2}}|\hat\varphi|_{1,\hat T}\le Ch^{\frac{1}{2}}\|\mu\|_{0,T}^{\frac{1}{2}}|\mu|_{1,T}^{\frac{1}{2}}|\varphi|_{1,T}\le Ch^{\frac{1}{2}}|u|_{2,T}^{\frac{1}{2}}|u|_{3,T}^{\frac{1}{2}}|w_h|_{2,T}.\end{aligned} \tag{4.12}$$

Furthermore, from (4.6) and (4.12) we conclude that the consistency error $E_{\varepsilon,h}(u,w_h)$ is bounded by

$$|E_{\varepsilon,h}(u,w_h)|\le C\varepsilon h^{\frac{1}{2}}|u|_{2,\Omega}^{\frac{1}{2}}|u|_{3,\Omega}^{\frac{1}{2}}|||w_h|||_{\varepsilon,h},$$

for any $w\in V_h$, Hence, by Lemma 2.2

$$|E_{\varepsilon,h}(u,w_h)|\le Ch^{\frac{1}{2}}\|f\|_{0,\Omega}|||w_h|||_{\varepsilon,h}, \tag{4.13}$$

and together with (4.4) and (4.9) this completes the proof. □

5. Conclusion

In this paper, using bubble functions, we construct a nonconforming C^0 cuboid element to solve the fourth order elliptic singular perturbation problem in three dimensions. The element is proved to be convergent uniformly respect to the perturbation parameter. Besides the theoretical interest, our new finite element method is expected to be useful in the computation of the Cahn-Hilliard equation. This will be our next work.

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