

ON SOLVABILITY AND WAVEFORM RELAXATION METHODS FOR LINEAR VARIABLE-COEFFICIENT DIFFERENTIAL-ALGEBRAIC EQUATIONS *

Xi Yang

*Department of Mathematics, Nanjing University of Aeronautics and Astronautics,
Nanjing 210016, China*

Email: yangxi@lsec.cc.ac.cn yangxi@nuaa.edu.cn

Abstract

This paper is concerned with the solvability and waveform relaxation methods of linear variable-coefficient differential-algebraic equations (DAEs). Most of the previous works have been focused on linear variable-coefficient DAEs with smooth coefficients and data, yet no results related to the convergence rate of the corresponding waveform relaxation methods has been obtained. In this paper, we develop the solvability theory for the linear variable-coefficient DAEs on Lebesgue square-integrable function space in both traditional and least squares senses, and determine the convergence rate of the waveform relaxation methods for solving linear variable-coefficient DAEs.

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1. Introduction

Consider the linear variable-coefficient differential-algebraic equations (also called singular systems, descriptor systems, implicit or constrained systems)

$$B(t)\dot{x}(t) + A(t)x(t) = f(t), \quad \forall t \in \Omega \subseteq \mathbb{R}, \quad (1.1)$$

where $B(t)$ and $A(t)$ are $r \times r$ complex matrix-valued coefficients, $x(t)$ and $f(t)$ are r -dimensional complex vector-valued functions, and $B(t)$ is identically singular. The theory for linear constant-coefficient DAEs has become well developed over the last 30 years; see [7, 8]. However, progress on the linear variable-coefficient DAEs has been less complete. The main idea for studying the existence and uniqueness of the solution of the DAEs (1.1) is utilizing coordinate changes to reduce the DAEs (1.1) to the so-called standard canonical form [20],

$$\begin{cases} \dot{y}(t) + C(t)y(t) = g(t), \\ N(t)\dot{z}(t) + z(t) = h(t), \end{cases}$$

where $N(t)$ is strictly lower triangular. This canonical form approach was continued in [16, 25]. However, examples in [6, 9, 10] showed that not all solvable systems could be put into this canonical form. The stability of the DAEs (1.1) is often studied in the sense of continuous dependence of the solution on initial value; see [5, 12]. The derivation of numerical methods for

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the DAEs (1.1) is closely related to determining all or part of the completion of the original vector field defined by the DAEs (1.1), i.e., determining the following ODEs derived from the DAEs (1.1),

$$\dot{x}(t) = Q(t)x(t) + \sum_{i=0}^{\ell} R_i(t)f^{(i)}(t). \quad (1.2)$$

The approximate solution computed by numerical methods related to the completion can be essentially arbitrary off the solution manifold if no restriction is placed on these numerical methods.

In the solvability theory of the DAEs (1.1) related to the standard canonical form, the variable-coefficients $B(t)$ and $A(t)$ are assumed $2r$ -times differentiable, data $f(t)$ is assumed at least r -times differentiable, and if $f(t)$ is assumed m -times differentiable, $r \leq m \leq 2r$, the solution $x(t)$ needs to be assumed $(m - r + 1)$ -times differentiable. In this paper, we study the solvability in a completely new way under weaker smoothness requirements, i.e., coefficients $B(t)$ and $A(t)$, data $f(t)$, and solution $x(t)$ are Lebesgue square-integrable. By using the Fourier transform, the linear variable-coefficient DAEs (1.1) are transformed into the Fredholm integral equation of the first kind, which is studied by taking advantage of the theory of compact operator. The solvability of the linear variable-coefficient DAEs (1.1) is discussed in both traditional and least squares senses. We eventually find the explicit expression of the solution and the sufficient conditions to guarantee existence and uniqueness of the solution. Furthermore, we are concerned with the stability of the DAEs (1.1) in the sense of the continuous dependence of the solution on data $f(t)$ rather than the initial value.

For numerical solution of the linear variable-coefficient DAEs (1.1), we concentrate on the waveform relaxation methods instead of the methods based on computing the completion (1.2). The waveform relaxation methods are powerful solvers for numerically computing the solution of the DAEs on both sequential and parallel computers; see [1–3, 19, 24, 26]. The basic idea of the above iteration methods is to apply the relaxation technique directly to the DAEs. Therefore, these methods can be regarded as natural extensions of the classical relaxation methods for solving systems of algebraic equations with iterating space changing from \mathbb{R}^r to the function space. The waveform relaxation methods were first introduced by Lelarasme in [18] for simulating the behavior of very large-scale electrical networks. Lelarasme proved that the waveform relaxation method is convergent as long as the splitting function of the system is Lipschitz continuous. Later, most of the effort has been made on the expansions and applications of this theory; see [15]. However, there is no precise description about the convergence rate until Miekkala first obtained the convergence rate of the waveform relaxation method of linear constant-coefficient ODEs and DAEs; see [21, 22]. Then, Janssen and Vandewalle studied the convergence rate of different SOR acceleration schemes of the waveform relaxation method for linear constant-coefficient ODEs in [14]. In addition, Pan and Bai further studied the monotone convergence rate of the waveform relaxation methods for linear constant-coefficient ODEs in [23]. The latest result related to the convergence rate of the waveform relaxation method of linear constant-coefficient DAEs is given by Bai and Yang in [4]. In this paper, we study the waveform relaxation methods for solving linear variable-coefficient DAEs in both traditional and least squares sense. The explicit iteration form of the waveform relaxation methods is first proposed. Moreover, we find the spectral radius of the iteration operator, i.e., convergence rate, of the waveform relaxation methods.

The paper is organized as follows. The theories of integral operators are generalized from

the scalar version to the matrix-type version in Section 2. The solvability of linear variable-coefficient DAEs (1.1) is discussed in both traditional and least squares senses in Section 3. Through the application of the theories developed in Section 3, the waveform relaxation methods are studied in Section 4. And the numerical results in Section 5 shows that the waveform relaxation methods are feasible and efficient for solving linear variable-coefficient DAEs.

2. Matrix-Type Integral Operators

In this section, we are concerned with the theory of matrix-type integral operators. Here and in the sequel, we denote $\mathbf{L}_2^r[0, 1]$ as the Hilbert space consisting of complex vector-valued functions with the inner product

$$(f(x), g(x)) = \int_0^1 \sum_{i=1}^r f_i(x) \overline{g_i(x)} dx, \quad \forall f(x), g(x) \in \mathbf{L}_2^r[0, 1],$$

where the integral is in the Lebesgue sense, and the corresponding norm is defined as

$$\|f(x)\| = \sqrt{(f(x), f(x))}, \quad \forall f(x) \in \mathbf{L}_2^r[0, 1].$$

And we denote $\mathbf{L}_2^{r \times r}[0, 1]^2$ as the Hilbert space consisting of complex matrix-valued functions with the inner product

$$(K(x, y), G(x, y)) = \sum_{i=1}^r \sum_{j=1}^r \int_0^1 \int_0^1 K_{ij}(x, y) \overline{G_{i,j}(x, y)} dx dy, \\ \forall K(x, y), G(x, y) \in \mathbf{L}_2^{r \times r}[0, 1]^2,$$

where the integral is in the Lebesgue sense, and the corresponding norm is defined as

$$\|K(x, y)\| = \sqrt{(K(x, y), K(x, y))}, \quad \forall K(x, y) \in \mathbf{L}_2^{r \times r}[0, 1]^2.$$

For convenience, we also denote

$$(u, v) = \sum_{i=1}^r u_i \overline{v_i}, \quad \forall u, v \in \mathbb{C}^r$$

as the inner product of the r -dimensional complex vector space \mathbb{C}^r , and the corresponding norm is defined as $\|u\| = \sqrt{(u, u)}, \forall u \in \mathbb{C}^r$.

The matrix-type integral operator \mathcal{K} with kernel $K(x, y) \in \mathbf{L}_2^{r \times r}[0, 1]^2$ is defined as

$$\mathcal{K}f(x) = \int_0^1 K(x, y) f(y) dy, \quad \forall f(x) \in \mathbf{L}_2^r[0, 1].$$

It is easy to verify that the adjoint of \mathcal{K} is of the form

$$\mathcal{K}^* f(x) = \int_0^1 K^*(x, y) f(y) dy, \quad \forall f(x) \in \mathbf{L}_2^r[0, 1].$$

Firstly, we prove a useful lemma.

Lemma 2.1. *Let $K(x, y) \in \mathbf{L}_2^{r \times r}[0, 1]^2$ be a matrix-valued kernel, then the associated matrix-type integral operator \mathcal{K} is bounded, i.e.*

$$\|\mathcal{K}f(x)\| \leq \|K(x, y)\| \|f(x)\|, \quad \forall f(x) \in \mathbf{L}_2^r[0, 1].$$

Proof. By direct computation, we have

$$\begin{aligned} \|\mathcal{K}f(x)\| &= \left\| \begin{pmatrix} \int_0^1 \sum_{j=1}^r K_{1j}(x, y) f_j(y) dy \\ \vdots \\ \int_0^1 \sum_{j=1}^r K_{rj}(x, y) f_j(y) dy \end{pmatrix} \right\| \\ &= \sqrt{\sum_{i=1}^r \int_0^1 dx \left| \int_0^1 \sum_{j=1}^r K_{ij}(x, y) f_j(y) dy \right|^2} \\ &\leq \sqrt{\sum_{i=1}^r \int_0^1 dx \left| \int_0^1 \sum_{j=1}^r |K_{ij}(x, y)| |f_j(y)| dy \right|^2} \\ &\leq \sqrt{\sum_{i=1}^r \int_0^1 dx \left| \int_0^1 \left(\sum_{j=1}^r |K_{ij}(x, y)|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^r |f_j(y)|^2 \right)^{\frac{1}{2}} dy \right|^2} \\ &\leq \sqrt{\sum_{i=1}^r \int_0^1 dx \int_0^1 \left(\sum_{j=1}^r |K_{ij}(x, y)|^2 \right) dy \int_0^1 \left(\sum_{j=1}^r |f_j(y)|^2 \right) dy} \\ &= \sqrt{\sum_{i=1}^r \sum_{j=1}^r \int_0^1 \int_0^1 |K_{i,j}(x, y)|^2 dx dy} \|f(x)\| \\ &= \|K(x, y)\| \|f(x)\|. \end{aligned}$$

Since $K(x, y) \in \mathbf{L}_2^{r \times r}[0, 1]^2$, $\|K(x, y)\|$ is a bounded constant. Therefore, \mathcal{K} is a bounded matrix-type integral operator. □

With the help of Lemma 2.1, we obtain the following theorem.

Theorem 2.1. *Let $K(x, y)$ be a $r \times r$ continuous matrix-valued kernel on $[0, 1]^2$. Then the associated matrix-type integral operator*

$$\mathcal{K}f(x) = \int_0^1 K(x, y) f(y) dy, \quad \forall f(x) \in \mathbf{L}_2^r[0, 1],$$

is a compact operator.

Proof. Since $K(x, y)$ is continuous on $[0, 1]^2$, then $K(x, y) \in \mathbf{L}_2^{r \times r}[0, 1]^2$. Based on Lemma 2.1, the operator \mathcal{K} is bounded, i.e.,

$$\|\mathcal{K}f(x)\| \leq \|K(x, y)\| \|f(x)\|.$$

Without loss of generality, we assume that $\|K(x, y)\| = 1$. Taking any uniformly bounded sequence $\{f^{(n)}(x)\}$, s.t. $\|f^{(n)}(x)\| \leq 1$, and considering the sequence $\{g^{(n)}(x)\} = \{\mathcal{K}f^{(n)}(x)\}$,

then we shall show that there is a convergent subsequence $\{g^{(n')}(x)\}$ that converges to a function in $\mathbf{L}_2^r[0, 1]$.

Denote $\{r_k\}$ as the dense countable set of all rational numbers in $[0, 1]$. Now considering the vector sequence $\{g^{(n)}(r_1)\} \subset \mathbb{C}^r$, direct computation leads to

$$\begin{aligned} \|g^{(n)}(r_1)\| &= \left\| \int_0^1 K(r_1, y) f^{(n)}(y) dy \right\| \\ &= \left(\sum_{i=1}^r \left| \int_0^1 \sum_{j=1}^r K_{ij}(r_1, y) f_j^{(n)}(y) dy \right|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^r \left| \int_0^1 \sum_{j=1}^r |K_{ij}(r_1, y)| |f_j^{(n)}(y)| dy \right|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^r \left| \int_0^1 \left(\sum_{j=1}^r |K_{ij}(r_1, y)|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^r |f_j^{(n)}(y)|^2 \right)^{\frac{1}{2}} dy \right|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^r \int_0^1 \left(\sum_{j=1}^r |K_{ij}(r_1, y)|^2 \right) dy \int_0^1 \left(\sum_{j=1}^r |f_j^{(n)}(y)|^2 \right) dy \right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^r \sum_{j=1}^r \int_0^1 |K_{ij}(r_1, y)|^2 dy \right)^{\frac{1}{2}} \|f^{(n)}(x)\| \\ &\leq \left(\sum_{i=1}^r \sum_{j=1}^r \int_0^1 |K_{ij}(r_1, y)|^2 dy \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, $\{g^{(n)}(r_1)\}$ is a uniformly bounded sequence in \mathbb{C}^r , and this leads to the existence of a subsequence $\{g^{(n,1)}(r_1)\}$ that converges to a vector $b^{(1)} \in \mathbb{C}^r$. Now we examine $\{g^{(n,1)}(r_1)\}$ and select a subsequence $\{g^{(n,2)}(r_2)\}$ that converges to a vector $b^{(2)} \in \mathbb{C}^r$. Repeat the process at all rational numbers. Finally we consider the sequence of functions $\{g^{(n,n)}(x)\}$ which converges at all rational numbers by virtue of its construction.

Reindex $\{g^{(n,n)}(x)\}$ as $\{g^{(n')}(x)\}$, and define function $g(x)$, s.t.,

$$g(r_k) := \lim_{n' \rightarrow \infty} g^{(n')}(r_k), \quad \forall k \in \mathbb{N}_+.$$

So that $g(x)$ is well-defined on $\{r_k\}$. We can also show that $g(x)$ is continuous on $\{r_k\}$. Based on the fact that $K(x, y)$ is continuous, we have

$$\begin{aligned} \forall \epsilon > 0, \exists \delta(\epsilon) > 0, \quad \forall |x_1 - x_2| < \delta(\epsilon), \quad x_1, x_2 \in \{r_k\} \text{ s.t.}, \\ \left(\sum_{i=1}^r \sum_{j=1}^r \int_0^1 |K_{ij}(x_1, y) - K_{ij}(x_2, y)|^2 dy \right)^{\frac{1}{2}} < \epsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} \|g(x_1) - g(x_2)\| &= \lim_{n' \rightarrow \infty} \|g^{(n')}(x_1) - g^{(n')}(x_2)\| \\ &= \lim_{n' \rightarrow \infty} \left\| \int_0^1 (K_{ij}(x_1, y) - K_{ij}(x_2, y)) f^{(n')}(y) dy \right\| \\ &\leq \left(\sum_{i=1}^r \sum_{j=1}^r \int_0^1 |K_{ij}(x_1, y) - K_{ij}(x_2, y)|^2 dy \right)^{\frac{1}{2}} < \epsilon. \end{aligned}$$

Now we define $g(x)$ for arbitrary $x \in [0, 1]$. Let $\{x_k\} \subset \{r_k\}$ be a rational sequence, s.t., $\lim_{k \rightarrow \infty} x_k = x$, we define

$$g(x) := \lim_{k \rightarrow \infty} g(x_k).$$

The above $g(x)$ is shown to be well-defined. Suppose that $\{\tilde{x}_k\} \subset \{r_k\}$ is another rational sequence converging to x , and suppose that $\lim_{k \rightarrow \infty} g(\tilde{x}_k) = \tilde{g}(x)$. Based on the definitions of $g(x)$ and $\tilde{g}(x)$, and the continuity of $g(x)$ on $\{r_k\}$, we have

$$\forall \epsilon > 0, \exists K, \forall k > K, \text{ s.t.}, \\ \|g(x) - g(x_k)\| < \frac{1}{3}\epsilon, \quad \|g(x_k) - g(\tilde{x}_k)\| < \frac{1}{3}\epsilon, \quad \|g(\tilde{x}_k) - \tilde{g}(x)\| < \frac{1}{3}\epsilon.$$

Therefore,

$$\|g(x) - \tilde{g}(x)\| \leq \|g(x) - g(x_k)\| + \|g(x_k) - g(\tilde{x}_k)\| + \|g(\tilde{x}_k) - \tilde{g}(x)\| < \epsilon.$$

It means that $g(x) = \tilde{g}(x)$. Furthermore, it is easy to show that $g(x)$ is continuous on $[0, 1]$, then we also have $g(x) \in \mathbf{L}^r[0, 1]$. It follows that \mathcal{K} is compact. \square

To facilitate our sequential discussion, we introduce a theorem concerned with compact operator as follows.

Theorem 2.2. ([13]) *Let \mathcal{K}_n be a sequence of compact operator on a Hilbert space \mathbf{H} , such that for some operator \mathcal{K} we have*

$$\lim_{n \rightarrow \infty} \|\mathcal{K} - \mathcal{K}_n\| = 0.$$

Then \mathcal{K} is also compact.

Based on Theorems 2.2 and 2.1 we obtain the following result with respect to matrix-type integral operator with $\mathbf{L}_2^{r \times r}[0, 1]^2$ kernels.

Theorem 2.3. *Let $K(x, y)$ be a matrix-valued kernel in $\mathbf{L}_2^{r \times r}[0, 1]^2$. Then the associated matrix-type integral operator*

$$\mathcal{K}f(x) = \int_0^1 K(x, y)f(y)dy, \quad \forall f(x) \in \mathbf{L}_2^r[0, 1],$$

is a compact operator.

Now we introduce an important result of compact operators on $\mathbf{L}_2^{r \times r}(\mathbb{R}^2)$. Let \mathcal{K} be a matrix-type integral operator with $\mathbf{L}_2^{r \times r}(\mathbb{R}^2)$ kernel $K(x, y)$. Denote $\{\mathcal{K}^{(n)}\}$ as a sequence of matrix-type integral operator with sequence of kernels $\{K^{(n)}(x, y)\}$ which are defined as

$$K^{(n)}(x, y) = \begin{cases} K(x, y), & -n \leq x, y \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Since any finite interval $[a, b]$ can be normalized to $[0, 1]$, previous Lemma 2.1 and Theorems 2.1 and 2.3 can be applied to $[a, b]$, such as $[-n, n]$ for any given positive integer n . Obviously, $\{\mathcal{K}^{(n)}\}$ is a sequence of compact operators on $\mathbf{L}_2^{r \times r}[-n, n]^2$. Hence, $\{\mathcal{K}^{(n)}\}$ can also be considered as a sequence of compact operators on $\mathbf{L}_2^{r \times r}(\mathbb{R}^2)$. Based on the fact that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|\mathcal{K} - \mathcal{K}^{(n)}\| \\ & \leq \lim_{n \rightarrow \infty} \sqrt{\sum_{i=1}^r \sum_{j=1}^r \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K_{i,j}(x, y) - K_{i,j}^{(n)}(x, y)|^2 dx dy} = 0 \end{aligned}$$

and Theorem 2.2, we have the following theorem.

Theorem 2.4. *Let $K(x, y)$ be a matrix-valued kernel in $\mathbf{L}_2^{r \times r}(\mathbb{R}^2)$. Then the associated matrix-type integral operator*

$$\mathcal{K}f(x) = \int_{-\infty}^{\infty} K(x, y)f(y)dy, \forall f(x) \in \mathbf{L}_2^r(\mathbb{R}),$$

is a compact operator.

3. Solvability of Linear Variable-Coefficient DAEs

In this section, we consider the solvability of the linear variable-coefficient DAEs (1.1) in complex vector-valued function space $\mathbf{L}_2^r(\mathbb{R})$ with the help of the Fourier transform. Due to the definition of space $\mathbf{L}_2^r(\mathbb{R})$, value of function on a single point is no longer essential. Hence, the initial (or boundary) condition of the DAEs (1.1) is ignored. For the application of the Fourier transform, we need to extend $B(t)$, $A(t)$, $f(t)$, and $x(t)$ to the whole real axis \mathbb{R} by defining

$$\xi(t) = \begin{cases} \tilde{\xi}(t) & t \in \Omega, \\ 0 & \text{otherwise,} \end{cases} \quad \text{here } \tilde{\xi}(t) = B(t), A(t), f(t), x(t),$$

and assuming the existence of the Fourier transform of $\xi(t)$.

Definition 3.1. ([13]) *An operator \mathcal{U} acting on a Hilbert space \mathbf{H} is said to be unitary if it is*

- *isometric, i.e., $\|\mathcal{U}f\| = \|f\|, \forall f \in \mathbf{H}$;*
- *\mathcal{U} has an inverse on all of \mathbf{H} given by \mathcal{U}^* , the adjoint of \mathcal{U} .*

We choose the kernel $\frac{1}{\sqrt{2\pi}}e^{-i\omega t}$ such that the Fourier transform of a function $f(t)$ is defined by

$$\mathcal{F}f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt = \tilde{f}(\omega),$$

then the above Fourier transform is a unitary operator mapping the space $\mathbf{L}_2^r(\mathbb{R})$ into itself; see [13]. Furthermore, we have the following lemma.

Lemma 3.1. ([11]) *Suppose that $\tilde{f}(\omega)$ and $\tilde{g}(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$ respectively, then the following properties are satisfied*

- $\mathcal{F}(f(t)g(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega')\tilde{g}(\omega - \omega')d\omega'$;
- $\mathcal{F}\dot{f}(t) = i\omega\tilde{f}(\omega)$.

3.1. Traditional Solution

By performing the Fourier transform on both sides of the linear variant-coefficient DAEs (1.1) and applying Lemma 3.1, we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega' \tilde{B}(\omega - \omega')\tilde{x}(\omega')d\omega' + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{A}(\omega - \omega')\tilde{x}(\omega')d\omega' = \tilde{f}(\omega),$$

which can be formulated as a Fredholm integral equation of the first kind

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left(i\omega' \tilde{B}(\omega - \omega') + \tilde{A}(\omega - \omega') \right) \tilde{x}(\omega')d\omega' = \tilde{f}(\omega). \tag{3.1}$$

Equivalently, Eq. (3.1) can be considered as a linear operator equation of the first kind,

$$\tilde{\mathcal{K}}\tilde{x} = \tilde{f}, \tag{3.2}$$

where $\tilde{\mathcal{K}}$ is a linear integral operator with kernel

$$\tilde{K}(\omega, \omega') = \frac{1}{\sqrt{2\pi}} \left(i\omega' \tilde{B}(\omega - \omega') + \tilde{A}(\omega - \omega') \right).$$

Denote $\tilde{\mathcal{K}}^*$ as the adjoint of operator $\tilde{\mathcal{K}}$, i.e.,

$$(\tilde{\mathcal{K}}^*\circ)(\omega') = \int_{-\infty}^{\infty} \tilde{K}^*(\omega, \omega') \circ d\omega.$$

Assume that the kernel $\tilde{K}(\omega, \omega') \in \mathbf{L}_2^{r \times r}(\mathbb{R}^2)$, according to Theorem 2.4, the corresponding operator $\tilde{\mathcal{K}}$ is a linear compact operator mapping $\mathbf{L}_2^r(\mathbb{R})$ into itself, so is the adjoint $\tilde{\mathcal{K}}^*$. Due to E. Schmidt’s theory of singular functions in [17], there exists a singular system for compact operator $\tilde{\mathcal{K}}$. In addition, since the Hilbert space $\mathbf{L}_2^r(\mathbb{R})$ is separable, all orthogonal subsets of $\mathbf{L}_2^r(\mathbb{R})$ are finite or countable. Therefore, the corresponding singular system can be presented as $\{\tilde{\sigma}_i, \tilde{v}_i, \tilde{u}_i\}_{i=1}^L$ such that

$$\tilde{\mathcal{K}}\tilde{v}_i = \tilde{\sigma}_i\tilde{u}_i, \quad \tilde{\mathcal{K}}^*\tilde{u}_i = \tilde{\sigma}_i\tilde{v}_i,$$

here L is a positive integer (finite or infinite), $\tilde{\sigma}_i > 0$ (if L is infinite, we have $\lim_{i \rightarrow \infty} \tilde{\sigma}_i = 0$), and $\tilde{v}_i, \tilde{u}_i \in \mathbf{L}_2^r(\mathbb{R})$. Furthermore, sequences $\{\tilde{v}_i\}$ and $\{\tilde{u}_i\}$ are both orthonormal sets in Hilbert space $\mathbf{L}_2^r(\mathbb{R})$, but not necessarily complete. Based on the above singular system, we have

$$\text{span}\{\tilde{u}_i\} = \overline{\text{range}(\tilde{\mathcal{K}})} = \text{null}(\tilde{\mathcal{K}}^*)^\perp, \quad \text{span}\{\tilde{v}_i\} = \overline{\text{range}(\tilde{\mathcal{K}}^*)} = \text{null}(\tilde{\mathcal{K}})^\perp,$$

which lead to the following orthogonal sums

$$\begin{aligned} \mathbf{L}_2^r(\mathbb{R}) &= \text{null}(\tilde{\mathcal{K}}^*)^\perp \oplus \text{null}(\tilde{\mathcal{K}}^*) = \overline{\text{range}(\tilde{\mathcal{K}})} \oplus \text{null}(\tilde{\mathcal{K}}^*) \\ &= \text{null}(\tilde{\mathcal{K}})^\perp \oplus \text{null}(\tilde{\mathcal{K}}) = \overline{\text{range}(\tilde{\mathcal{K}}^*)} \oplus \text{null}(\tilde{\mathcal{K}}). \end{aligned}$$

In general, the range of $\tilde{\mathcal{K}}$ and that of $\tilde{\mathcal{K}}^*$ are not closed, i.e., $\text{range}(\tilde{\mathcal{K}}) \subset \overline{\text{range}(\tilde{\mathcal{K}})}$ and $\text{range}(\tilde{\mathcal{K}}^*) \subset \overline{\text{range}(\tilde{\mathcal{K}}^*)}$ in most cases.

Denote \mathcal{P}_1 as the orthogonal projector of $\mathbf{L}_2^r(\mathbb{R})$ onto $\text{null}(\tilde{\mathcal{K}})$. Any given vector $\phi \in \mathbf{L}_2^r(\mathbb{R})$ can be written into the unique orthogonal decomposition

$$\phi = \phi^\perp + \phi^{(0)} = (\mathcal{I} - \mathcal{P}_1)\phi + \mathcal{P}_1\phi \in \text{null}(\tilde{\mathcal{K}})^\perp \oplus \text{null}(\tilde{\mathcal{K}}).$$

Therefore, the acting of operator $\tilde{\mathcal{K}}$ on ϕ is as follows

$$\begin{aligned} \tilde{\mathcal{K}} : \quad \mathbf{L}_2^r(\mathbb{R}) &\rightarrow \text{range}(\tilde{\mathcal{K}}) \\ \phi = \phi^\perp + \phi^{(0)} &\mapsto \tilde{\mathcal{K}}(\phi^\perp). \end{aligned}$$

Denote $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}})$ as the quotient space of $\mathbf{L}_2^r(\mathbb{R})$ mod $\text{null}(\tilde{\mathcal{K}})$, and $\phi^{(Q)} \in \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}})$ is of the form

$$\phi^{(Q)} = \phi^\perp + \text{null}(\tilde{\mathcal{K}}), \text{ here } \phi^\perp \in \text{null}(\tilde{\mathcal{K}})^\perp.$$

Furthermore, any $\phi^{(a)} \in \phi^{(Q)}$ can be considered as a representative of $\phi^{(Q)}$, thus $\phi^{(a)}$ and $\phi^{(Q)}$ can be considered as the same.

Denote \mathcal{P}_2 as the orthogonal projector of $\mathbf{L}_2^r(\mathbb{R})$ onto $\text{null}(\tilde{\mathcal{K}}^*)$. Any given vector $\psi \in \mathbf{L}_2^r(\mathbb{R})$ can be written into the unique orthogonal decomposition

$$\psi = \psi^\perp + \psi^{(0)} = (\mathcal{I} - \mathcal{P}_2)\psi + \mathcal{P}_2\psi \in \text{null}(\tilde{\mathcal{K}}^*)^\perp \oplus \text{null}(\tilde{\mathcal{K}}^*).$$

Therefore, the acting of operator $\tilde{\mathcal{K}}^*$ on ψ is as follows

$$\begin{aligned} \tilde{\mathcal{K}}^* : \quad \mathbf{L}_2^r(\mathbb{R}) &\rightarrow \text{range}(\tilde{\mathcal{K}}^*) \\ \psi = \psi^\perp + \psi^{(0)} &\mapsto \tilde{\mathcal{K}}^*(\psi^\perp). \end{aligned}$$

Denote $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*)$ as the quotient space of $\mathbf{L}_2^r(\mathbb{R})$ mod $\text{null}(\tilde{\mathcal{K}}^*)$, and $\psi^{(Q)} \in \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*)$ is of the form

$$\psi^{(Q)} = \psi^\perp + \text{null}(\tilde{\mathcal{K}}^*), \text{ here } \psi^\perp \in \text{null}(\tilde{\mathcal{K}}^*)^\perp.$$

Furthermore, any $\psi^{(a)} \in \psi^{(Q)}$ can be considered as a representative of $\psi^{(Q)}$, thus $\psi^{(a)}$ and $\psi^{(Q)}$ can be considered as the same.

Since $\text{null}(\tilde{\mathcal{K}})$ and $\text{null}(\tilde{\mathcal{K}}^*)$ are invariant subspaces of $\mathbf{L}_2^r(\mathbb{R})$ for operator $\tilde{\mathcal{K}}$ and its adjoint $\tilde{\mathcal{K}}^*$ respectively, there is a natural interpretation of $\tilde{\mathcal{K}}$ as an operator with respect to quotient spaces, i.e.,

$$\begin{aligned} \tilde{\mathcal{K}}^{(Q)} : \quad \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}) &\rightarrow (\text{range}(\tilde{\mathcal{K}}) + \text{null}(\tilde{\mathcal{K}}^*))/\text{null}(\tilde{\mathcal{K}}^*) \subseteq \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*) \\ \phi^{(Q)} = \phi^\perp + \text{null}(\tilde{\mathcal{K}}) &\mapsto \psi^{(Q)} = \tilde{\mathcal{K}}(\phi^\perp) + \text{null}(\tilde{\mathcal{K}}^*) \\ \text{or equivalently } \phi^{(a)} &\mapsto \psi^{(a)}. \end{aligned}$$

Obviously, operator $\tilde{\mathcal{K}}^{(Q)}$ is bijective and invertible. In this manner, the linear operator Eq. (3.2) can be considered as the following linear operator equation of the first kind,

$$\tilde{\mathcal{K}}^{(Q)}\tilde{x} = \tilde{f}, \tag{3.3}$$

here \tilde{x} and \tilde{f} are treated as representatives of the corresponding vectors in quotient spaces $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}})$ and $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*)$ respectively.

Similarly, a natural interpretation of $\tilde{\mathcal{K}}^*$ as a operator with respect to quotient spaces can be demonstrated as

$$\begin{aligned} \tilde{\mathcal{K}}^{*(Q)} : \quad & \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*) \rightarrow (\text{range}(\tilde{\mathcal{K}}^*) + \text{null}(\tilde{\mathcal{K}}))/\text{null}(\tilde{\mathcal{K}}) \subseteq \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}) \\ \psi^{(Q)} = \psi^\perp + \text{null}(\tilde{\mathcal{K}}^*) \mapsto & \phi^{(Q)} = \tilde{\mathcal{K}}^*(\psi^\perp) + \text{null}(\tilde{\mathcal{K}}) \\ \text{or equivalently } \psi^{(q)} \mapsto & \phi^{(q)}, \end{aligned}$$

and operator $\tilde{\mathcal{K}}^{*(Q)}$ is again bijective and invertible.

The natural definitions of inner products for $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}})$ is as

$$(\phi_1^{(q)}, \phi_2^{(q)})_q := (\phi_1^\perp, \phi_2^\perp), \quad \forall \phi_1^{(q)}, \phi_2^{(q)} \in \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}),$$

the corresponding norm is defined as

$$\|\phi^{(q)}\|_q = \sqrt{(\phi^{(q)}, \phi^{(q)})_q}, \quad \forall \phi^{(q)} \in \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}),$$

and for $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*)$ is as

$$(\psi_1^{(q)}, \psi_2^{(q)})_q := (\psi_1^\perp, \psi_2^\perp), \quad \forall \psi_1^{(q)}, \psi_2^{(q)} \in \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*),$$

the corresponding norm is defined as $\|\psi^{(q)}\|_q = \sqrt{(\psi^{(q)}, \psi^{(q)})_q}$, $\forall \psi^{(q)} \in \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*)$. The quotient spaces $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}})$ and $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*)$ are Hilbert spaces with respect to the above definitions of inner products.

Since quotient spaces $(\text{range}(\tilde{\mathcal{K}}) + \text{null}(\tilde{\mathcal{K}}^*))/\text{null}(\tilde{\mathcal{K}}^*)$ and $(\text{range}(\tilde{\mathcal{K}}^*) + \text{null}(\tilde{\mathcal{K}}))/\text{null}(\tilde{\mathcal{K}})$ are not necessarily complete, the inverse operators $(\tilde{\mathcal{K}}^{(Q)})^{-1}$ and $(\tilde{\mathcal{K}}^{*(Q)})^{-1}$ are not necessarily bounded.

Due to the linearity and compactness of operator $\tilde{\mathcal{K}}$ and its adjoint $\tilde{\mathcal{K}}^*$, operator $\tilde{\mathcal{K}}^{(Q)}$ and its adjoint $\tilde{\mathcal{K}}^{*(Q)}$ are both linear compact operators. The corresponding singular system of operator $\tilde{\mathcal{K}}^{(Q)}$ is $\{\tilde{\sigma}_i^{(q)}, \tilde{v}_i^{(q)}, \tilde{u}_i^{(q)}\}_{i=1}^L$, here

$$\tilde{\sigma}_i^{(q)} = \tilde{\sigma}_i, \quad \tilde{v}_i^{(q)} \in \tilde{v}_i + \text{null}(\tilde{\mathcal{K}}), \quad \text{and} \quad \tilde{u}_i^{(q)} \in \tilde{u}_i + \text{null}(\tilde{\mathcal{K}}^*).$$

Furthermore, sequences $\{\tilde{v}_i^{(q)}\}$ and $\{\tilde{u}_i^{(q)}\}$ are complete orthonormal sets in Hilbert spaces $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}})$ and $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*)$ respectively. Based on the above singular system, we obtain the explicit expression of the operator $\tilde{\mathcal{K}}^{(Q)}$,

$$\tilde{\mathcal{K}}^{(Q)}(\circ) = \sum_{k=1}^L \tilde{\sigma}_k \tilde{u}_k^{(q)}(\circ, \tilde{v}_k^{(q)})_q,$$

which can be verified as

$$\begin{aligned} \tilde{\mathcal{K}}^{(Q)}\tilde{v}_i^{(q)} &= \sum_{k=1}^L \tilde{\sigma}_k \tilde{u}_k^{(q)}(\tilde{v}_i^{(q)}, \tilde{v}_k^{(q)})_q = \tilde{\sigma}_i \tilde{u}_i^{(q)}(\tilde{v}_i^{(q)}, \tilde{v}_i^{(q)})_q \\ &= \tilde{\sigma}_i \tilde{u}_i^{(q)}(\tilde{v}_i, \tilde{v}_i) = \tilde{\sigma}_i \tilde{u}_i^{(q)}, \\ \tilde{\mathcal{K}}^{*(Q)}\tilde{u}_i^{(q)} &= \sum_{k=1}^L \tilde{\sigma}_k \tilde{v}_k^{(q)}(\tilde{u}_i^{(q)}, \tilde{u}_k^{(q)})_q = \tilde{\sigma}_i \tilde{v}_i^{(q)}(\tilde{u}_i^{(q)}, \tilde{u}_i^{(q)})_q \\ &= \tilde{\sigma}_i \tilde{v}_i^{(q)}(\tilde{u}_i, \tilde{u}_i) = \tilde{\sigma}_i \tilde{v}_i^{(q)}. \end{aligned}$$

Since sequences $\{\tilde{v}_i^{(q)}\}$ and $\{\tilde{u}_i^{(q)}\}$ are complete orthonormal sets, $\tilde{x} \in \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}})$ and $\tilde{f} \in \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*)$ can be written into the following unique linear combinations respectively,

$$\tilde{x} = \sum_{i=1}^L \alpha_i \tilde{v}_i^{(q)}, \quad \tilde{f} = \sum_{i=1}^L (\tilde{f}, \tilde{u}_i^{(q)})_q \tilde{u}_i^{(q)}.$$

In addition,

$$\tilde{f} = \tilde{\mathcal{K}}^{(Q)} \tilde{x} = \tilde{\mathcal{K}}^{(Q)} \sum_{i=1}^L \alpha_i \tilde{v}_i^{(q)} = \sum_{i=1}^L \alpha_i \tilde{\sigma}_i \tilde{u}_i^{(q)},$$

due to the uniqueness of the linear combination of \tilde{f} , we have

$$\alpha_i \tilde{\sigma}_i = (\tilde{f}, \tilde{u}_i^{(q)})_q = (\tilde{f}, \tilde{u}_i) \Rightarrow \alpha_i = \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i}.$$

Hence

$$\|\tilde{x}\| = \sum_{i=1}^L \left| \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \right|^2.$$

In order to guarantee the existence of $\tilde{x} \in \mathbf{L}_2^r(\mathbb{R})$, we assume that

$$\sum_{i=1}^L \left| \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \right|^2 < \infty, \quad \text{i.e.,} \quad \left\{ \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \right\} \in l_2.$$

By direct verification of Eq. (3.2), we have

$$\begin{aligned} \tilde{\mathcal{K}} \tilde{x} &= \tilde{\mathcal{K}} ((\mathcal{I} - \mathcal{P}_1) \tilde{x} + \mathcal{P}_1 \tilde{x}) \\ &= \tilde{\mathcal{K}} \left(\sum_{i=1}^L \alpha_i (\mathcal{I} - \mathcal{P}_1) v_i^{(q)} + \sum_{i=1}^L \alpha_i \mathcal{P}_1 \tilde{v}_i^{(q)} \right) \\ &= \tilde{\mathcal{K}} \left(\sum_{i=1}^L \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} v_i + \sum_{i=1}^L \alpha_i \mathcal{P}_1 \tilde{v}_i^{(q)} \right) \\ &= \sum_{i=1}^L (\tilde{f}, \tilde{u}_i) \tilde{u}_i = (\mathcal{I} - \mathcal{P}_2) \tilde{f}. \end{aligned}$$

Therefore, we need the condition $(\mathcal{I} - \mathcal{P}_2) \tilde{f} = \tilde{f}$ to guarantee the satisfaction of Eq. (3.2), i.e., condition $\tilde{f} \in \text{null}(\tilde{\mathcal{K}}^*)^\perp$.

Based on the above analysis, we see that

$$\begin{aligned} \tilde{x} &= (\mathcal{I} - \mathcal{P}_1) \tilde{x} + \mathcal{P}_1 \tilde{x} \\ &= \sum_{i=1}^L \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} v_i + \sum_{i=1}^L \alpha_i \mathcal{P}_1 \tilde{v}_i^{(q)} \end{aligned}$$

is a vector in $\mathbf{L}_2^r(\mathbb{R})$ solves Eq. (3.2). Since the set of all solutions of the linear operator Eq. (3.2) is closed and convex, then there is a unique solution with the smallest norm of the form

$$(\mathcal{I} - \mathcal{P}_1) \tilde{x} = \sum_{i=1}^L \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} v_i.$$

Meanwhile, \tilde{x} as a representative of a vector in quotient space $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}})$ solves Eq. (3.3).

Since $\tilde{\mathcal{K}}^{(Q)}$ is bijective, we have

$$\tilde{x} = (\tilde{\mathcal{K}}^{(Q)})^{-1}\tilde{f} = \sum_{i=1}^L \tilde{v}_i^{(q)} \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i}. \tag{3.4}$$

i.e., $(\tilde{\mathcal{K}}^{(Q)})^{-1}$ has the following explicit expression

$$(\tilde{\mathcal{K}}^{(Q)})^{-1}(\circ) = \sum_{i=1}^L \tilde{v}_i^{(q)} \frac{(\circ, \tilde{u}_i)}{\tilde{\sigma}_i}. \tag{3.5}$$

By acting the inverse Fourier transform \mathcal{F}^{-1} on both sides of Eq. (3.4), we obtain the explicit expression of the solution of the linear variable-coefficient DAEs (1.1)

$$x = \mathcal{F}^{-1}(\tilde{\mathcal{K}}^{(Q)})^{-1}\mathcal{F}f \in \mathbf{L}_2^r(\mathbb{R}),$$

and the corresponding solution with the smallest norm

$$x = \mathcal{F}^{-1}(\mathcal{I} - \mathcal{P}_1)(\tilde{\mathcal{K}}^{(Q)})^{-1}\mathcal{F}f.$$

Suppose there is a perturbation on \tilde{f} , say, $\tilde{f}_\epsilon = \tilde{f} + \epsilon\tilde{u}_n^{(q)}$, $\epsilon > 0$, then the perturbation on \tilde{x} is as the following form

$$\begin{aligned} \|\tilde{x}_\epsilon - \tilde{x}\|_q &= \|(\tilde{\mathcal{K}}^{(Q)})^{-1}(\tilde{f}_\epsilon - \tilde{f})\|_q \\ &= \epsilon \|(\tilde{\mathcal{K}}^{(Q)})^{-1}\tilde{u}_n^{(q)}\|_q \\ &= \epsilon \left\| \frac{\tilde{v}_n^{(q)}}{\tilde{\sigma}_n} \right\|_q = \frac{\epsilon}{\tilde{\sigma}_n}. \end{aligned}$$

If the singular system is infinite, i.e., $L = \infty$, we have

$$\lim_{n \rightarrow \infty} \|\tilde{x}_\epsilon - \tilde{x}\|_q = \lim_{n \rightarrow \infty} \frac{\epsilon}{\tilde{\sigma}_n} = \infty,$$

which leads to the fact that operator $(\tilde{\mathcal{K}}^{(Q)})^{-1}$ is unbounded, or equivalently discontinuous. Therefore, Eq. (3.3) is ill-posed¹⁾, which means that the solution of the linear variable-coefficient DAEs (1.1) with the smallest norm is an ill-posed problem. If the singular system is finite, i.e., $L < \infty$, we have

$$\lim_{\epsilon \rightarrow 0} \|\tilde{x}_\epsilon - \tilde{x}\|_q = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\tilde{\sigma}_n} = 0, \quad \forall n = 1, 2, \dots, L,$$

this means operator $(\tilde{\mathcal{K}}^{(Q)})^{-1}$ is continuous. Therefore, Eq. (3.3) is well-posed, which means that the solution of the linear variable-coefficient DAEs (1.1) with the smallest norm is a well-posed problem.

Now we summarize all the previous results in the form of a theorem.

¹⁾ The notion of an ill-posed problem, or more correctly that of a well-posed problem, was introduced by Hadamard more than a century ago. Within the context of the theory of partial differential equations Hadamard termed a problem well-posed if it has a solution (existence), it does not have more than one solution (uniqueness), and this solution depends continuously on the data of the problem (stability). The last of the previous three conditions is motivated by the fact that in applications the data will be measured quantities and therefore always contaminated by errors.

Theorem 3.1. Consider the linear variable-coefficient DAEs (1.1) on Hilbert space $\mathbf{L}_2^r(\mathbb{R})$. Assume that the kernel $\tilde{K}(\omega, \omega') \in \mathbf{L}_2^{r \times r}(\mathbb{R}^2)$, $\tilde{f} \in \text{null}(\tilde{K}^*)^\perp$ and $\left\{ \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \right\} \in l_2$, then the solution of the DAEs (1.1) is of the following form

$$x = \mathcal{F}^{-1}(\tilde{K}^{(Q)})^{-1} \mathcal{F} f \in \mathbf{L}_2^r(\mathbb{R}), \tag{3.6}$$

and the unique solution with the smallest norm is

$$x = \mathcal{F}^{-1}(\mathcal{I} - \mathcal{P}_1)(\tilde{K}^{(Q)})^{-1} \mathcal{F} f. \tag{3.7}$$

Furthermore, if L is infinite, the DAEs (1.1) is ill-posed; if L is finite, the DAEs (1.1) is well-posed.

Remark 3.1. We end this subsection with two remarks.

- In fact, the linear system of the DAEs (1.1) is analogous to the linear system of algebraic equations in the sense of linearity and compatibility. Specifically, the solution (3.6) of the DAEs (1.1) is analogous to the solution of compatible linear system of algebraic equations, and the solution (3.7) of the DAEs (1.1) with the smallest norm is analogous to the solution of compatible linear system of algebraic equations with the smallest norm.
- The quantity L in Theorem 3.1 is imposed for classification of compact operator. If L is finite, the compact operator \tilde{K} is degenerate. Similar to the operator $\tilde{K}^{(Q)}$, the explicit expression of the operator \tilde{K} is of the form

$$\begin{aligned} \tilde{K}(\circ) &= \sum_{k=1}^L \tilde{\sigma}_k \tilde{u}_k(\circ, \tilde{v}_k) \\ &= \sum_{k=1}^L \tilde{\sigma}_k \tilde{u}_k(x) \int_0^\infty \tilde{v}_k^*(y) \circ dy \\ &= \int_0^\infty \sum_{k=1}^L \tilde{\sigma}_k \tilde{u}_k(x) \tilde{v}_k^*(y) \circ dy, \end{aligned}$$

i.e., \tilde{K} is an integral operator with finite sum kernel $\tilde{K}(x, y) = \sum_{k=1}^L \tilde{\sigma}_k \tilde{u}_k(x) \tilde{v}_k^*(y)$, which is called a degenerate kernel. In this case, the corresponding linear operator equation of the first kind is well-posed. If L is infinite, we obtain an ill-posed linear operator equation of the first kind. There are great challenges with numerical solution of problem of this kind.

3.2. Least Squares Solution

Due to the Theorem 3.1, a solution of the form (3.6) of the linear variable-coefficient DAEs (1.1) exists if the Fourier transform of data $f(t)$ satisfies $\tilde{f} \in \text{null}(\tilde{K}^*)^\perp$ and $\left\{ \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \right\} \in l_2$. However, this is not always the case in many applications. Hence there is a demand to broaden our notion of solution, which can be done by enlarging the class of data $f(t)$ such that a type of least squares solution exists. A vector $x \in \mathbf{L}_2^r(\mathbb{R})$ is called a least squares solution of the linear variable-coefficient DAEs (1.1) if

$$\|B(t)\dot{x}(t) + A(t)x(t) - f(t)\| = \inf_{y \in \mathbf{L}_2^r(\mathbb{R})} \|B(t)\dot{y}(t) + A(t)y(t) - f(t)\|. \tag{3.8}$$

Since the Fourier transform is unitary, performing the Fourier transform on both sides of Eq. (3.8) leads to the following equivalent least squares problem,

$$\|\tilde{\mathcal{K}}\tilde{x} - \tilde{f}\| = \inf_{\tilde{y} \in \mathbf{L}_2^r(\mathbb{R})} \|\tilde{\mathcal{K}}\tilde{y} - \tilde{f}\|, \tag{3.9}$$

where $\tilde{\mathcal{K}}$ is defined as the same in Subsection 3.1. Assume that the kernel $\tilde{K}(\omega, \omega') \in \mathbf{L}_2^{r \times r}(\mathbb{R}^2)$, then the corresponding operator $\tilde{\mathcal{K}}$ is a linear compact operator mapping $\mathbf{L}_2^r(\mathbb{R})$ into itself with singular system $\{\tilde{\sigma}_i, \tilde{v}_i, \tilde{u}_i\}_{i=1}^L$ defined as the same in Subsection 3.1.

Since any given $\tilde{f} \in \mathbf{L}_2^r(\mathbb{R})$ can be written into the unique orthogonal decomposition as the following

$$\tilde{f} = \tilde{f}^\perp + \tilde{f}^{(0)} = (\mathcal{I} - \mathcal{P}_2)\tilde{f} + \mathcal{P}_2\tilde{f} \in \text{null}(\tilde{\mathcal{K}}^*)^\perp \oplus \text{null}(\tilde{\mathcal{K}}^*),$$

then \tilde{f} and \tilde{f}^\perp both can be considered as the representative of a same vector in quotient space $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}^*)$. In addition, the least squares problem (3.9) is equivalent to the following linear operator equation of the first kind,

$$\tilde{\mathcal{K}}\tilde{x} = \tilde{f}^\perp = (\mathcal{I} - \mathcal{P}_2)\tilde{f} \in \text{null}(\tilde{\mathcal{K}}^*)^\perp.$$

By assuming that $\sum_{i=1}^L \left| \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \right|^2 < \infty$, i.e., $\left\{ \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \right\} \in l_2$, similar to the analysis in Subsection 3.1, we have

$$\begin{aligned} \tilde{x} &= (\tilde{\mathcal{K}}^{(Q)})^{-1}\tilde{f}^\perp = \sum_{i=1}^L \tilde{v}_i^{(q)} \frac{(\tilde{f}^\perp, \tilde{u}_i^{(q)})_q}{\tilde{\sigma}_i} \\ &= \sum_{i=1}^L \tilde{v}_i^{(q)} \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} = \sum_{i=1}^L \tilde{v}_i^{(q)} \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \\ &= \sum_{i=1}^L \tilde{v}_i^{(q)} \frac{(\tilde{f}, \tilde{u}_i^{(q)})_q}{\tilde{\sigma}_i} = (\tilde{\mathcal{K}}^{(Q)})^{-1}\tilde{f}, \end{aligned}$$

which is a solution of the least squares problem (3.9). By acting the inverse Fourier transform, we obtain a solution of the least squares problem (3.8) as the following

$$x = \mathcal{F}^{-1}(\tilde{\mathcal{K}}^{(Q)})^{-1}\mathcal{F}f.$$

Now we consider the uniqueness of solution, note that the least squares problem (3.9) is equivalent to

$$\tilde{\mathcal{K}}\tilde{x} - \tilde{f} \in \text{null}(\tilde{\mathcal{K}}^*)^\perp,$$

i.e.,

$$\tilde{\mathcal{K}}^*\tilde{\mathcal{K}}\tilde{x} = \tilde{\mathcal{K}}^*\tilde{f}.$$

Based on the above linear operator equation, we see that there is a unique least squares solution if and only if

$$\{0\} = \text{null}(\tilde{\mathcal{K}}^*\tilde{\mathcal{K}}).$$

Nevertheless, the above condition is not satisfied in most cases. Note that the set of all least squares solutions is closed and convex, hence there is a unique least squares solution with the smallest norm of the following form

$$(\mathcal{I} - \mathcal{P}_1)\tilde{x} = (\mathcal{I} - \mathcal{P}_1)(\tilde{\mathcal{K}}^{(Q)})^{-1}\tilde{f} = \sum_{i=1}^L \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} v_i. \tag{3.10}$$

Obviously, the operator $(\mathcal{I} - \mathcal{P}_1)(\tilde{\mathcal{K}}^{(Q)})^{-1}$ associates a given function $\tilde{f} \in \text{range}(\tilde{\mathcal{K}}) + \text{null}(\tilde{\mathcal{K}}^*) \subseteq \mathbf{L}_2^r(\mathbb{R})$ with the unique least squares solution which has the smallest norm. We denote this operator by $\tilde{\mathcal{K}}^\dagger$, which is defined as

$$\begin{aligned} \tilde{\mathcal{K}}^\dagger : \quad \text{range}(\tilde{\mathcal{K}}) + \text{null}(\tilde{\mathcal{K}}^*) \subseteq \mathbf{L}_2^r(\mathbb{R}) &\rightarrow \text{null}(\tilde{\mathcal{K}})^\perp \\ \tilde{f} &\mapsto \tilde{\mathcal{K}}^\dagger \tilde{f} = (\mathcal{I} - \mathcal{P}_1)(\tilde{\mathcal{K}}^{(Q)})^{-1}\tilde{f}. \end{aligned}$$

The operator $\tilde{\mathcal{K}}^\dagger$ is called the Moore-Penrose generalized inverse of $\tilde{\mathcal{K}}$, which is an analog to the Moore-Penrose generalized inverse of a matrix. By acting the inverse Fourier transform, we obtain the solution with the smallest norm of the least squares problem (3.8) as the following

$$x = \mathcal{F}^{-1}\tilde{\mathcal{K}}^\dagger \mathcal{F}f.$$

According to Eq. (3.10), the operator $\tilde{\mathcal{K}}^\dagger$ can be written explicitly as

$$\tilde{\mathcal{K}}^\dagger(\circ) = \sum_{i=1}^L \tilde{v}_i \frac{(\circ, \tilde{u}_i)}{\tilde{\sigma}_i}. \tag{3.11}$$

As we argued in Subsection 3.1, the operator $\tilde{\mathcal{K}}^\dagger$ is continuous if L is finite. Then, the least squares problem (3.9) is well-posed, which means that the solution of the least squares problem (3.8) with the smallest norm is a well-posed problem. In addition, the operator $\tilde{\mathcal{K}}^\dagger$ is discontinuous if L is infinite. Then, the least squares problem (3.9) is ill-posed, which means that the solution of the least squares problem (3.8) with the smallest norm is an ill-posed problem.

In summary, we have the following theorem.

Theorem 3.2. *Consider the least squares solution of the linear variable-coefficient DAEs (1.1) on Hilbert space $\mathbf{L}_2^r(\mathbb{R})$. If the kernel $\tilde{K}(\omega, \omega') \in \mathbf{L}_2^{r \times r}(\mathbb{R}^2)$ and $\left\{ \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \right\} \in l_2$, then the solution of the least squares problem (3.8) is of the following form*

$$x = \mathcal{F}^{-1}(\tilde{\mathcal{K}}^{(Q)})^{-1}\mathcal{F}f \in \mathbf{L}_2^r(\mathbb{R}), \tag{3.12}$$

and the unique least squares solution with the smallest norm is

$$x = \mathcal{F}^{-1}\tilde{\mathcal{K}}^\dagger \mathcal{F}f. \tag{3.13}$$

Furthermore, if L is infinite, the solution of the least squares problem (3.8) is ill-posed; if L is finite, the solution of the least squares problem (3.8) is well-posed.

4. The Waveform Relaxation Methods

In this section, we discuss the waveform relaxation methods for the linear variable-coefficient DAEs (1.1) in the sense of traditional solution and least squares solution.

4.1. The Waveform Relaxation Methods in the Sense of Traditional Solution

For the linear variable-coefficient DAEs (1.1), we split the variable-coefficients $B(t)$ and $A(t)$ into

$$B(t) = M_B(t) - N_B(t), \quad A(t) = M_A(t) - N_A(t),$$

respectively. Then the corresponding waveform relaxation method is defined as

$$\begin{aligned} & M_B(t)\dot{x}^{(k+1)}(t) + M_A(t)x^{(k+1)}(t) \\ & = N_B(t)\dot{x}^{(k)}(t) + N_A(t)x^{(k)}(t) + f(t), \quad \forall t \in \Omega \subseteq \mathbb{R}. \end{aligned} \tag{4.1}$$

Since we choose function space $\mathbf{L}_2^r(\mathbb{R})$ for our discussion of the above waveform relaxation method (4.1), the initial condition of the method in (4.1) is no longer vital. For the application of the Fourier transform, we extend $M_B(t)$, $N_B(t)$, $M_A(t)$, $N_A(t)$, $f(t)$ and $x^{(k)}(t)$ to the whole real axis \mathbb{R} by defining

$$\xi(t) = \begin{cases} \tilde{\xi}(t) & t \in \Omega, \\ 0 & \text{otherwise,} \end{cases} \quad \text{here } \tilde{\xi}(t) = M_B(t), N_B(t), M_A(t), N_A(t), f(t), x^{(k)}(t),$$

and assuming the existence of the Fourier transform of $\xi(t)$.

By performing the Fourier transform on both sides of the waveform relaxation method (4.1) and applying Lemma 3.1, we obtain a Fredholm integral equation of the first kind

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left(i\omega' \tilde{M}_B(\omega - \omega') + \tilde{M}_A(\omega - \omega') \right) \tilde{x}^{(k+1)}(\omega') d\omega' \\ & = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left(i\omega' \tilde{N}_B(\omega - \omega') + \tilde{N}_A(\omega - \omega') \right) \tilde{x}^{(k)}(\omega') d\omega' + \tilde{f}(\omega). \end{aligned} \tag{4.2}$$

Equivalently, the above integral equation can be written into the following operator iteration scheme

$$\tilde{\mathcal{K}}_M \tilde{x}^{(k+1)} = \tilde{\mathcal{K}}_N \tilde{x}^{(k)} + \tilde{f}, \tag{4.3}$$

where $\tilde{\mathcal{K}}_M$ and $\tilde{\mathcal{K}}_N$ are linear integral operators with kernels

$$\begin{aligned} \tilde{K}_M(\omega, \omega') &= \frac{1}{\sqrt{2\pi}} \left(i\omega' \tilde{M}_B(\omega - \omega') + \tilde{M}_A(\omega - \omega') \right), \\ \tilde{K}_N(\omega, \omega') &= \frac{1}{\sqrt{2\pi}} \left(i\omega' \tilde{N}_B(\omega - \omega') + \tilde{N}_A(\omega - \omega') \right), \end{aligned}$$

respectively. It is easy to verify that

$$\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_M - \tilde{\mathcal{K}}_N,$$

i.e., the iteration scheme (4.3) is a splitting iterative method of the linear operator Eq. (3.2) in Subsection 3.2.

Assume that the kernels $\tilde{K}_M(\omega, \omega'), \tilde{K}_N(\omega, \omega') \in \mathbf{L}_2^r(\mathbb{R}^2)$, then the linear integral operators $\tilde{\mathcal{K}}_M$ and $\tilde{\mathcal{K}}_N$ are compact on Hilbert space $\mathbf{L}_2^r(\mathbb{R})$. Therefore, there exists a singular system of operator $\tilde{\mathcal{K}}_M$, $\{\tilde{\sigma}_{i,M}, \tilde{v}_{i,M}, \tilde{u}_{i,M}\}_{i=1}^{L_M}$ such that

$$\tilde{\mathcal{K}}_M \tilde{v}_{i,M} = \tilde{\sigma}_{i,M} \tilde{u}_{i,M}, \quad \tilde{\mathcal{K}}_M^* \tilde{u}_{i,M} = \tilde{\sigma}_{i,M} \tilde{v}_{i,M},$$

here L_M is a positive integer (finite or infinite). Furthermore, sequences $\{\tilde{v}_{i,M}\}$ and $\{\tilde{u}_{i,M}\}$ are both orthonormal sets in Hilbert space $\mathbf{L}_2^r(\mathbb{R})$, but not necessarily complete. Based on the above singular system, we have the following orthogonal sums

$$\begin{aligned} \mathbf{L}_2^r(\mathbb{R}) &= \text{null}(\tilde{\mathcal{K}}_M^*)^\perp \oplus \text{null}(\tilde{\mathcal{K}}_M) = \overline{\text{range}(\tilde{\mathcal{K}}_M)} \oplus \text{null}(\tilde{\mathcal{K}}_M^*) \\ &= \text{null}(\tilde{\mathcal{K}}_M)^\perp \oplus \text{null}(\tilde{\mathcal{K}}_M) = \overline{\text{range}(\tilde{\mathcal{K}}_M^*)} \oplus \text{null}(\tilde{\mathcal{K}}_M). \end{aligned}$$

Assume that $\text{range}(\tilde{\mathcal{K}}_N) \subseteq \text{range}(\tilde{\mathcal{K}}_M)$, then $\tilde{\mathcal{K}}_N \tilde{x}^{(k)} \in \text{range}(\tilde{\mathcal{K}}_M) \subseteq \text{null}(\tilde{\mathcal{K}}_M^*)^\perp$, and there exists a function $\varphi \in \mathbf{L}_2^r(\mathbb{R})$ such that

$$\tilde{\mathcal{K}}_M \varphi = \tilde{\mathcal{K}}_N \tilde{x}^{(k)},$$

where φ can be explicitly written as

$$\begin{aligned} \varphi &= (\mathcal{I} - \mathcal{P}_{1,M})\varphi + \mathcal{P}_{1,M}\varphi \\ &= \sum_{i=1}^{L_M} \frac{(\tilde{\mathcal{K}}_N \tilde{x}^{(k)}, \tilde{u}_{i,M})}{\tilde{\sigma}_{i,M}} \tilde{v}_{i,M} + \mathcal{P}_{1,M}\varphi, \end{aligned}$$

where $\mathcal{P}_{1,M}$ denotes the orthogonal projector of $\mathbf{L}_2^r(\mathbb{R})$ onto $\text{null}(\tilde{\mathcal{K}}_M)$. Hence, we have

$$\left\| \left\{ \frac{(\tilde{\mathcal{K}}_N \tilde{x}^{(k)}, \tilde{u}_{i,M})}{\tilde{\sigma}_{i,M}} \right\} \right\| = \|(\mathcal{I} - \mathcal{P}_{1,M})\varphi\| \leq \|\varphi\| \leq \infty,$$

i.e., $\left\{ \frac{(\tilde{\mathcal{K}}_N \tilde{x}^{(k)}, \tilde{u}_{i,M})}{\tilde{\sigma}_{i,M}} \right\} \in l_2$. Assume that $\tilde{f} \in \text{null}(\tilde{\mathcal{K}}_M^*)^\perp$ and $\left\{ \frac{(\tilde{f}, \tilde{u}_{i,M})}{\tilde{\sigma}_{i,M}} \right\} \in l_2$, then we obtain

$$\tilde{\mathcal{K}}_N \tilde{x}^{(k)} + \tilde{f} \in \text{null}(\tilde{\mathcal{K}}_M^*)^\perp$$

and $\left\{ \frac{(\tilde{\mathcal{K}}_N \tilde{x}^{(k)} + \tilde{f}, \tilde{u}_{i,M})}{\tilde{\sigma}_{i,M}} \right\} \in l_2$. According to Theorem 3.1, the operator Eq. (4.3) is solvable. Therefore, the iteration scheme (4.3) is well-defined.

Similar to Subsection 3.1, we define the following bijective operator with respect to quotient spaces $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}_M)$ and $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}_M^*)$,

$$\begin{aligned} \tilde{\mathcal{K}}_M^{(Q)} : \quad \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}_M) &\rightarrow (\text{range}(\tilde{\mathcal{K}}_M) + \text{null}(\tilde{\mathcal{K}}_M^*))/\text{null}(\tilde{\mathcal{K}}_M^*) \subseteq \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}_M^*) \\ \phi^{(Q)} = \phi^\perp + \text{null}(\tilde{\mathcal{K}}_M) &\mapsto \psi^{(Q)} = \tilde{\mathcal{K}}_M(\phi^\perp) + \text{null}(\tilde{\mathcal{K}}_M^*) \\ \text{or equivalently } \phi^{(q)} &\mapsto \psi^{(q)}, \end{aligned}$$

and the adjoint of $\tilde{\mathcal{K}}_M^{(Q)}$ is defined accordingly,

$$\begin{aligned} \tilde{\mathcal{K}}_M^{*(Q)} : \quad \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}_M^*) &\rightarrow (\text{range}(\tilde{\mathcal{K}}_M^*) + \text{null}(\tilde{\mathcal{K}}_M))/\text{null}(\tilde{\mathcal{K}}_M) \subseteq \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}_M) \\ \psi^{(Q)} = \psi^\perp + \text{null}(\tilde{\mathcal{K}}_M^*) &\mapsto \phi^{(Q)} = \tilde{\mathcal{K}}_M^*(\psi^\perp) + \text{null}(\tilde{\mathcal{K}}_M) \\ \text{or equivalently } \psi^{(q)} &\mapsto \phi^{(q)}. \end{aligned}$$

The inner product for quotient spaces is defined in the same way as in Subsection 3.1, for $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}_M)$,

$$(\phi_1^{(q)}, \phi_2^{(q)})_q := (\phi_1^\perp, \phi_2^\perp), \quad \forall \phi_1^{(q)}, \phi_2^{(q)} \in \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}_M),$$

and for $\mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}_M^*)$,

$$(\psi_1^{(q)}, \psi_2^{(q)})_q := (\psi_1^\perp, \psi_2^\perp), \quad \forall \psi_1^{(q)}, \psi_2^{(q)} \in \mathbf{L}_2^r(\mathbb{R})/\text{null}(\tilde{\mathcal{K}}_M^*).$$

Furthermore, the corresponding singular system of operator $\tilde{\mathcal{K}}_M^{(Q)}$ is denoted as $\{\tilde{\sigma}_{i,M}, \tilde{v}_{i,M}^{(q)}, \tilde{u}_{i,M}^{(q)}\}_{i=1}^{L_M}$.

We consider $\{\tilde{x}^{(k)}\}$ as the sequence generated by the operator equation in the iteration scheme (4.3) with the smallest norm, hence we have

$$\tilde{x}^{(k+1)} = \tilde{\mathcal{G}}\tilde{x}^{(k)} + \tilde{\Phi}\tilde{f}, \quad (4.4)$$

where $\tilde{\mathcal{G}} = (\mathcal{I} - \mathcal{P}_{1,M})(\tilde{\mathcal{K}}_M^{(Q)})^{-1}\tilde{\mathcal{K}}_N$ and $\tilde{\Phi} = (\mathcal{I} - \mathcal{P}_{1,M})(\tilde{\mathcal{K}}_M^{(Q)})^{-1}$, or equivalently,

$$x^{(k+1)} = \mathcal{G}x^{(k)} + \Phi f,$$

where $\mathcal{G} = \mathcal{F}^{-1}(\mathcal{I} - \mathcal{P}_{1,M})(\tilde{\mathcal{K}}_M^{(Q)})^{-1}\tilde{\mathcal{K}}_N\mathcal{F}$ and $\Phi = \mathcal{F}^{-1}(\mathcal{I} - \mathcal{P}_{1,M})(\tilde{\mathcal{K}}_M^{(Q)})^{-1}\mathcal{F}$. Since operators $\mathcal{I} - \mathcal{P}_{1,M}$ and $\tilde{\mathcal{K}}_N$ are bounded, the iteration operator $\tilde{\mathcal{G}}$ in (4.4) is bounded if and only if $(\tilde{\mathcal{K}}_M^{(Q)})^{-1}$ is bounded, which is equivalent to $\tilde{\mathcal{K}}_M^{(Q)}$ being degenerate, i.e., $\tilde{\mathcal{K}}_M$ being degenerate. Therefore, the integer L_M needs to be finite.

Now we discuss the convergence property of the iteration scheme (4.3) with the assumption that $L_M < \infty$. By defining $\varepsilon^{(k+1)} = \tilde{x}^{(k+1)} - \tilde{x}^{(k)}$ and applying (3.5), we obtain

$$\begin{aligned} \varepsilon^{(k+1)} &= \tilde{\mathcal{G}}\varepsilon^{(k)} \\ &= (\mathcal{I} - \mathcal{P}_{1,M}) \sum_{j=1}^{L_M} \tilde{v}_{i,M}^{(q)} \frac{(\tilde{\mathcal{K}}_N \varepsilon^{(k)}, \tilde{u}_{j,M})}{\tilde{\sigma}_{j,M}} \\ &= \sum_{j=1}^{L_M} \tilde{v}_{j,M} \frac{(\tilde{\mathcal{K}}_N \varepsilon^{(k)}, \tilde{u}_{j,M})}{\tilde{\sigma}_{j,M}}. \end{aligned}$$

Due to the fact that $\{\tilde{x}^{(k)}\} \subset \text{null}(\tilde{\mathcal{K}}_M)^\perp$, each error function $\varepsilon^{(k)}$ can be uniquely expanded by sequence $\{\tilde{v}_{i,M}\}$, i.e., there exists a unique column vector $\alpha^{(k)} = (\alpha_1^{(k)}, \dots, \alpha_{L_M}^{(k)})^T$ such that

$$\varepsilon^{(k)} = \tilde{V}_M \alpha^{(k)} = \sum_{i=1}^{L_M} \alpha_i^{(k)} \tilde{v}_{i,M},$$

here $\tilde{V}_M = (\tilde{v}_{1,M}, \dots, \tilde{v}_{L_M,M})$, thus we have

$$\begin{aligned} \sum_{j=1}^{L_M} \alpha_j^{(k+1)} \tilde{v}_{j,M} &= \sum_{j=1}^{L_M} \tilde{v}_{j,M} \frac{(\tilde{\mathcal{K}}_N \sum_{i=1}^{L_M} \alpha_i^{(k)} \tilde{v}_{i,M}, \tilde{u}_{j,M})}{\tilde{\sigma}_{j,M}} \\ &= \sum_{j=1}^{L_M} \tilde{v}_{j,M} \frac{\sum_{i=1}^{L_M} \alpha_i^{(k)} (\tilde{\mathcal{K}}_N \tilde{v}_{i,M}, \tilde{u}_{j,M})}{\tilde{\sigma}_{j,M}}, \end{aligned}$$

By comparing the coefficient of $\tilde{v}_{j,M}$ on both sides, we have

$$\alpha^{(k+1)} = G^T \alpha^{(k)},$$

where $G = (g_{i,j})_{L_M \times L_M}$, $g_{i,j} = \frac{(\tilde{\mathcal{K}}_N \tilde{v}_{i,M}, \tilde{u}_{j,M})}{\tilde{\sigma}_{j,M}}$. Moreover, we have

$$\varepsilon^{(k+1)} = \tilde{V}_M \alpha^{(k+1)} = \tilde{V}_M G^T \alpha^{(k)} = \tilde{V}_M G^T \tilde{V}_M^* \tilde{V}_M \alpha^{(k)} = \tilde{V}_M G^T \tilde{V}_M^* \varepsilon^{(k)},$$

where \tilde{V}_M^* is defined as

$$\tilde{V}_M^* = \begin{pmatrix} (\circ, \tilde{v}_{1,M}) \\ \vdots \\ (\circ, \tilde{v}_{L_M,M}) \end{pmatrix}.$$

Therefore, we obtain the decomposition of the iteration operator in (4.4), i.e., $\tilde{\mathcal{G}} = \tilde{V}_M G^T \tilde{V}_M^*$. Furthermore, we can prove that $\rho(\tilde{\mathcal{G}}) = \rho(G)$. To see this, let $(\lambda, v) = (\lambda, \tilde{V}_M \beta)$ be an eigenpair of iteration operator $\tilde{\mathcal{G}}$, here $\beta = (\beta_1, \dots, \beta_{L_M})^T$, then we have

$$\tilde{V}_M G^T \tilde{V}_M^* v = \lambda v \Leftrightarrow \tilde{V}_M (G^T - \lambda I) \tilde{V}_M^* v = 0 \Leftrightarrow \tilde{V}_M (G^T - \lambda I) \beta = 0 \Leftrightarrow (G^T - \lambda I) \beta = 0,$$

this means that $\tilde{\mathcal{G}}$ and G^T have the same spectral set, which leads to the same spectral radius. Moreover, since the Fourier transform is unitary, we have $\rho(\tilde{\mathcal{G}}) = \rho(G^T) = \rho(G)$.

Then we obtain the following theorem addressing the convergence property of the waveform relaxation method (4.1).

Theorem 4.1. *Consider the linear variable-coefficient DAEs (1.1) on Hilbert space $\mathbf{L}_2^r(\mathbb{R})$ with the facts that the kernel $\tilde{K}(\omega, \omega') \in \mathbf{L}_2^{r \times r}(\mathbb{R}^2)$, $\tilde{f} \in \text{null}(\tilde{K}^*)^\perp$ and $\left\{ \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \right\} \in l_2$. Let*

$$B(t) = M_B(t) - N_B(t), \quad A(t) = M_A(t) - N_A(t),$$

be splittings of variable-coefficients $B(t)$ and $A(t)$ such that the kernels $\tilde{K}_M(\omega, \omega'), \tilde{K}_N(\omega, \omega') \in \mathbf{L}_2^r(\mathbb{R}^2)$, $\text{range}(\tilde{K}_N) \subseteq \text{range}(\tilde{K}_M)$, $\tilde{f} \in \text{null}(\tilde{K}_M^)^\perp$ and $\left\{ \frac{(\tilde{f}, \tilde{u}_{i,M})}{\tilde{\sigma}_{i,M}} \right\} \in l_2$, then the waveform relaxation method (4.1) can be written into the explicit operator form*

$$x^{(k+1)} = \mathcal{G}x^{(k)} + \Phi f,$$

where $\mathcal{G} = \mathcal{F}^{-1}(\mathcal{I} - \mathcal{P}_{1,M})(\tilde{K}_M^{(Q)})^{-1} \tilde{K}_N \mathcal{F}$ and $\Phi = \mathcal{F}^{-1}(\mathcal{I} - \mathcal{P}_{1,M})(\tilde{K}_M^{(Q)})^{-1} \mathcal{F}$. Furthermore, if the operator \tilde{K}_M is degenerate, i.e., $L_M < \infty$, the spectral radius of the iteration operator is $\rho(\mathcal{G}) = \rho(G)$.

4.2. The Waveform Relaxation Methods in the Sense of Least Squares Solution

In order to derive the waveform relaxation methods in the sense of least squares solution, we consider the least squares problem (3.9) which is equivalent to the following linear operator equation of the first kind,

$$\tilde{K}^* \tilde{K} \tilde{x} = \tilde{K}^* \tilde{f}.$$

Assume that $\text{range}(\tilde{K}_M) = \text{range}(\tilde{K})$, then we have $\text{null}(\tilde{K}_M^*) = \text{null}(\tilde{K}^*)$, which leads to

$$\tilde{K}_M^* \tilde{K} \tilde{x} = \tilde{K}_M^* \tilde{f}.$$

By substituting the splitting of operator $\tilde{K} = \tilde{K}_M - \tilde{K}_N$ into the above equation, we obtain

$$\tilde{K}_M^* (\tilde{K}_M - \tilde{K}_N) \tilde{x} = \tilde{K}_M^* \tilde{f}.$$

Thus we can define the following iteration scheme,

$$\begin{aligned} \tilde{K}_M^* \tilde{K}_M \tilde{x}^{(k+1)} &= \tilde{K}_M^* \tilde{K}_N \tilde{x}^{(k)} + \tilde{K}_M^* \tilde{f} \\ &= \tilde{K}_M^* (\tilde{K}_N \tilde{x}^{(k)} + \tilde{f}), \end{aligned}$$

which is equivalent to

$$\|\tilde{\mathcal{K}}_M \tilde{x}^{(k+1)} - (\tilde{\mathcal{K}}_N \tilde{x}^{(k)} + \tilde{f})\| = \inf_{\tilde{y} \in \mathbf{L}_2^r(\mathbb{R})} \|\tilde{\mathcal{K}}_M \tilde{y} - (\tilde{\mathcal{K}}_N \tilde{x}^{(k)} + \tilde{f})\|, \tag{4.5}$$

or equivalently,

$$\begin{aligned} & \left\| M_B(t) \dot{x}^{(k+1)}(t) + M_A(t) x^{(k+1)}(t) - (N_B(t) \dot{x}^{(k)}(t) + N_A(t) x^{(k)}(t) + f(t)) \right\| \\ &= \inf_{y \in \mathbf{L}_2^r(\mathbb{R})} \left\| M_B(t) \dot{y}(t) + M_A(t) y(t) - (N_B(t) \dot{x}^{(k)}(t) + N_A(t) x^{(k)}(t) + f(t)) \right\|. \end{aligned} \tag{4.6}$$

Assume that the kernels $\tilde{K}_M(\omega, \omega'), \tilde{K}_N(\omega, \omega') \in \mathbf{L}_2^r(\mathbb{R}^2)$, then the linear integral operators $\tilde{\mathcal{K}}_M$ and $\tilde{\mathcal{K}}_N$ are compact on Hilbert space $\mathbf{L}_2^r(\mathbb{R})$. Assume that $\left\{ \frac{(\tilde{f}, \tilde{u}_{i,M})}{\tilde{\sigma}_{i,M}} \right\} \in l_2$, together with the fact $\text{range}(\tilde{\mathcal{K}}_N) \subseteq \text{range}(\tilde{\mathcal{K}}_M)$, then we can prove that $\left\{ \frac{(\tilde{\mathcal{K}}_N \tilde{x}^{(k)} + \tilde{f}, \tilde{u}_{i,M})}{\tilde{\sigma}_{i,M}} \right\} \in l_2$. According to the Theorem 3.2, the least squares problem in (4.5) is solvable. Therefore, the iteration scheme (4.5) is well-defined.

We consider $\{\tilde{x}^{(k)}\}$ as the sequence generated by the least squares problem in the iteration scheme (4.5) with the smallest norm, hence we have

$$\tilde{x}^{(k+1)} = \tilde{\mathcal{G}} \tilde{x}^{(k)} + \tilde{\Phi} \tilde{f}, \tag{4.7}$$

here $\tilde{\mathcal{G}} = \tilde{\mathcal{K}}_M^\dagger \tilde{\mathcal{K}}_N$ and $\tilde{\Phi} = \tilde{\mathcal{K}}_M^\dagger$, or equivalently,

$$x^{(k+1)} = \mathcal{G} x^{(k)} + \Phi f,$$

here $\mathcal{G} = \mathcal{F}^{-1} \tilde{\mathcal{K}}_M^\dagger \tilde{\mathcal{K}}_N \mathcal{F}$ and $\Phi = \mathcal{F}^{-1} \tilde{\mathcal{K}}_M^\dagger \mathcal{F}$. The iteration operator $\tilde{\mathcal{G}}$ in (4.7) is bounded if and only if $\tilde{\mathcal{K}}_M^\dagger$ is bounded, which is equivalent to $\tilde{\mathcal{K}}_M$ being degenerate. Therefore, the integer L_M needs to be finite.

Now we discuss the convergence of the iteration scheme (4.5) with the assumption that $L_M < \infty$. By applying (3.11), we obtain

$$\varepsilon^{(k+1)} = \tilde{\mathcal{G}} \varepsilon^{(k)} = \sum_{j=1}^{L_M} \tilde{v}_{j,M} \frac{(\tilde{\mathcal{K}}_N \varepsilon^{(k)}, \tilde{u}_{j,M})}{\tilde{\sigma}_{j,M}}.$$

By denoting $\alpha^{(k)} = (\alpha_1^{(k)}, \dots, \alpha_{L_M}^{(k)})^T$ as the unique column vector such that

$$\varepsilon^{(k)} = \tilde{V}_M \alpha^{(k)} = \sum_{i=1}^{L_M} \alpha_i^{(k)} \tilde{v}_{i,M},$$

here $\tilde{V}_M = (\tilde{v}_{1,M}, \dots, \tilde{v}_{L_M,M})$. Similar to the Subsection 4.1, we have

$$\alpha^{(k+1)} = G^T \alpha^{(k)},$$

where $G = (g_{i,j})_{L_M \times L_M}$, $g_{i,j} = \frac{(\tilde{\mathcal{K}}_N \tilde{v}_{i,M}, \tilde{u}_{j,M})}{\tilde{\sigma}_{j,M}}$, we can prove that $\rho(\tilde{\mathcal{G}}) = \rho(G)$.

Thus we summarize the following convergence theorem of the waveform relaxation method (4.6).

Theorem 4.2. Consider the least squares solution of the linear variable-coefficient DAEs (1.1) on Hilbert space $\mathbf{L}_2^r(\mathbb{R})$ with the facts that the kernel $\tilde{K}(\omega, \omega') \in \mathbf{L}_2^{r \times r}(\mathbb{R}^2)$ and $\left\{ \frac{(\tilde{f}, \tilde{u}_i)}{\tilde{\sigma}_i} \right\} \in l_2$. Let

$$B(t) = M_B(t) - N_B(t), \quad A(t) = M_A(t) - N_A(t),$$

be splittings of variable-coefficients $B(t)$ and $A(t)$ such that the kernels $\tilde{K}_M(\omega, \omega'), \tilde{K}_N(\omega, \omega') \in \mathbf{L}_2^r(\mathbb{R}^2)$, $\text{range}(\tilde{K}_M) = \text{range}(\tilde{K})$ and $\left\{ \frac{(\tilde{f}, \tilde{u}_{i,M})}{\tilde{\sigma}_{i,M}} \right\} \in l_2$, then the waveform relaxation method (4.6) can be written into the explicit operator form

$$x^{(k+1)} = \mathcal{G}x^{(k)} + \Phi f,$$

here $\mathcal{G} = \mathcal{F}^{-1} \tilde{K}_M^\dagger \tilde{K}_N \mathcal{F}$ and $\Phi = \mathcal{F}^{-1} \tilde{K}_M^\dagger \mathcal{F}$. Furthermore, if the operator \tilde{K}_M is degenerate, i.e., $L_M < \infty$, the spectral radius of the iteration operator is $\rho(\mathcal{G}) = \rho(G)$.

5. Numerical Results

In actual computation, it is infeasible to perform an infinite time simulation on computer. Therefore, we choose a small linear system of differential-algebraic equations with variable coefficients, and take long time simulation to illustrate the behavior of waveform relaxation method.

Consider the initial value problem

$$\begin{aligned} & \begin{pmatrix} \frac{a t}{e^{a t}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} \frac{2}{e^{a t}} & \frac{\sin(b t)}{e^{a t}} \\ \frac{\sin(b t)}{e^{a t}} & \frac{2}{e^{a t}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ & = \begin{pmatrix} \frac{2}{e^{2 a t}} - \frac{a^2 t}{e^{2 a t}} + \frac{\sin(b t)^2}{e^{a t}(a^2 t^2 + 1)} \\ \frac{\sin(b t)}{e^{2 a t}} + \frac{2 \sin(b t)}{e^{a t}(a^2 t^2 + 1)} \end{pmatrix}, \end{aligned} \quad (5.1)$$

with the initial conditions $x_1(0) = 1$ and $x_2(0) = 0$. Here, the real part of the constant a is positive, and the constant b is real. In our tests, b is chosen to be $b = 1$. The exact solutions of (5.1) are $x_1^*(t) = e^{-a t}$ and $x_2^*(t) = \frac{\sin(b t)}{a^2 t^2 + 1}$. The single SOR waveform relaxation (SSORWR) method is employed to solve this problem, i.e., the splittings of the corresponding coefficients are given by

$$\begin{aligned} M_B &= \begin{pmatrix} \frac{a t}{e^{a t}} & 0 \\ 0 & 0 \end{pmatrix}, \quad N_B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ M_A &= \frac{1}{\tau} \begin{pmatrix} \frac{2}{e^{a t}} & 0 \\ 0 & \frac{2}{e^{a t}} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -\frac{\sin(b t)}{e^{a t}} & 0 \end{pmatrix}, \\ N_A &= \frac{1 - \tau}{\tau} \begin{pmatrix} \frac{2}{e^{a t}} & 0 \\ 0 & \frac{2}{e^{a t}} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{\sin(b t)}{e^{a t}} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

We solve the DAEs (5.1) on time interval $\Omega_t = [0, T]$, where $T = \Delta t \times \ell_t$, $\Delta t = 0.01$ is the time stepsize and ℓ_t represents the number of time steps. Here, we choose the parameter $\tau = 1$, and the constant $a = \frac{1}{T}$. In Figs. 5.1 and 5.2, we plot the error $\epsilon_2^{(k)}$ of the second component of the iterate after 0, 1, 2 and 3 iteration steps for $\ell_t = 5000$ and $\ell_t = 80000$, respectively. In both figures, we see that the errors decrease rapidly along the whole time interval Ω_t during the iteration process. Moreover, The upper bound of the error decreases monotonously after every

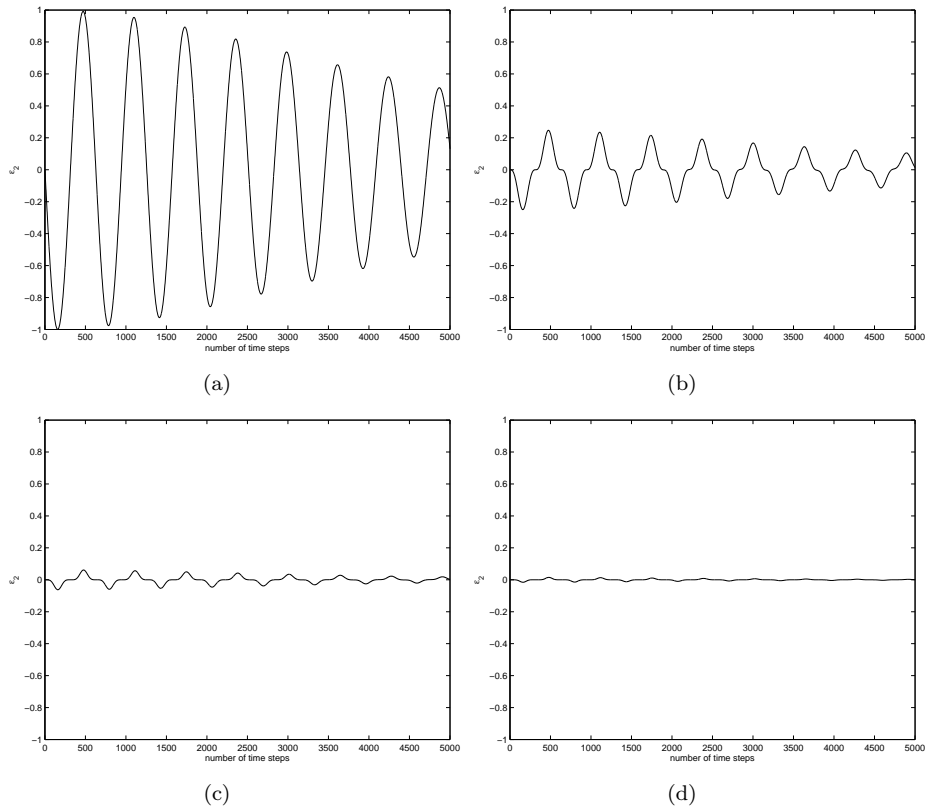


Fig. 5.1. The error $\epsilon_2^{(k)}$ after k iterations for 5000 time steps and $\tau = 1$: (a) $k = 0$; (b) $k = 1$; (c) $k = 2$; (d) $k = 3$.

Table 5.1: Number of iterations of the SSORWR method for different τ and time intervals.

<i>NTS</i>	5000	10000	20000	40000	80000
Interval	[0, 50.0]	[0, 100.0]	[0, 200.0]	[0, 400.0]	[0, 800.0]
$\tau = 0.25$	93	93	93	93	93
$\tau = 0.50$	40	40	40	40	40
$\tau = 0.75$	22	22	22	22	22
$\tau = 1.00$	11	11	11	11	11
$\tau = 1.25$	11	11	11	11	11
$\tau = 1.50$	22	22	22	22	22
$\tau = 1.75$	57	59	59	59	59

iteration. We remark that the situation for $\epsilon_1^{(k)}$ of the first component of the iterate is similar. Hence, we can expect that the SSORWR method is convergent on an infinite time interval.

In Table 5.1, we show the number of iterations of the SSORWR method based on different τ 's and different time intervals. The stopping criterion is defined as

$$\sup_{t \in \Omega_t} \left\{ |\epsilon_1^{(k)}|, |\epsilon_2^{(k)}| \right\} < 10^{-6}.$$

The SSORWR method converges for all the different cases. We see that the number of iterations of the SSORWR method keeps all most the same while the length of time interval increases 15 times for different τ 's, i.e., the length increases from 50 to 800. Fig. 5.3 shows the

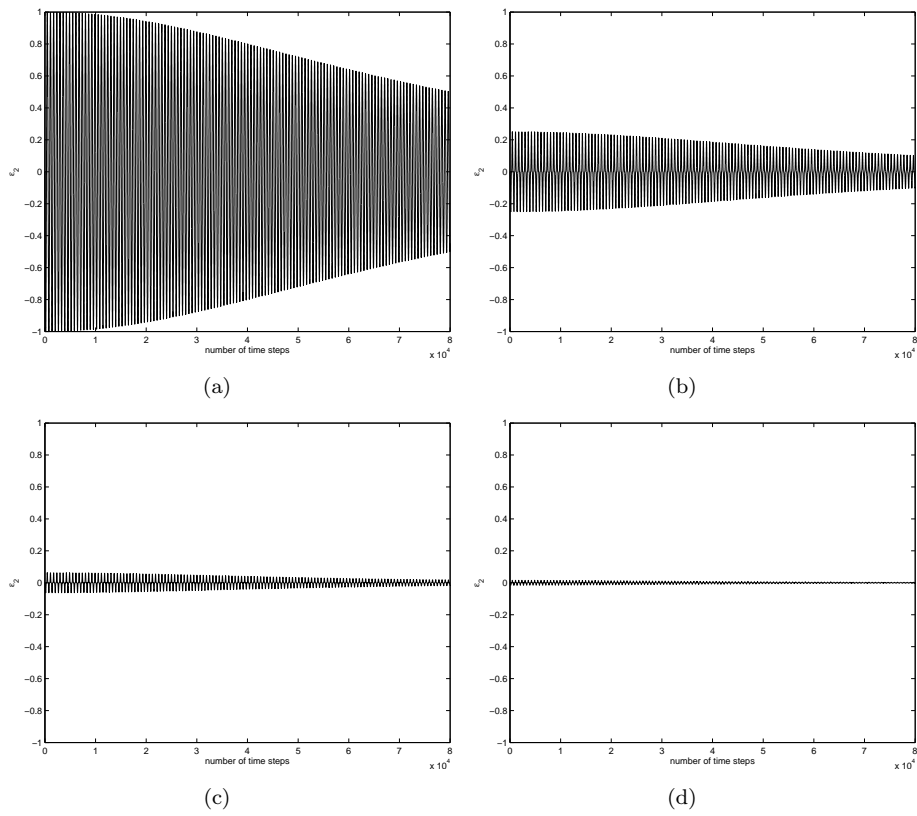


Fig. 5.2. The error $\epsilon_2^{(k)}$ after k iterations for 80000 time steps and $\tau = 1$: (a) $k = 0$; (b) $k = 1$; (c) $k = 2$; (d) $k = 3$.

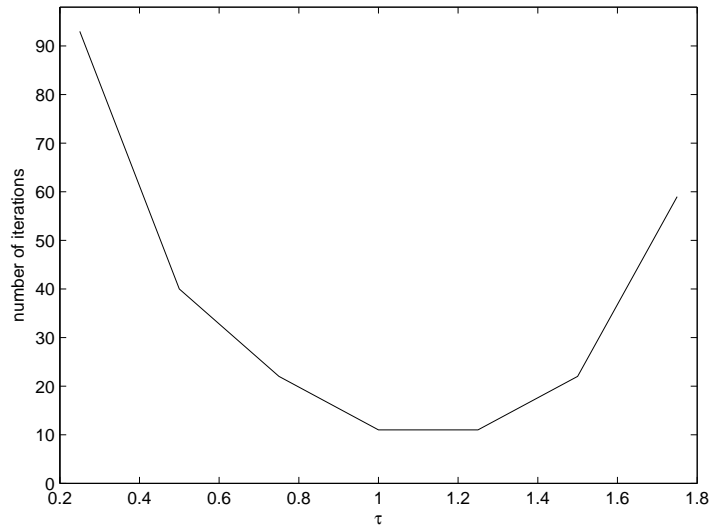


Fig. 5.3. Number of iterations versa parameter τ in the case of 20000 time steps.

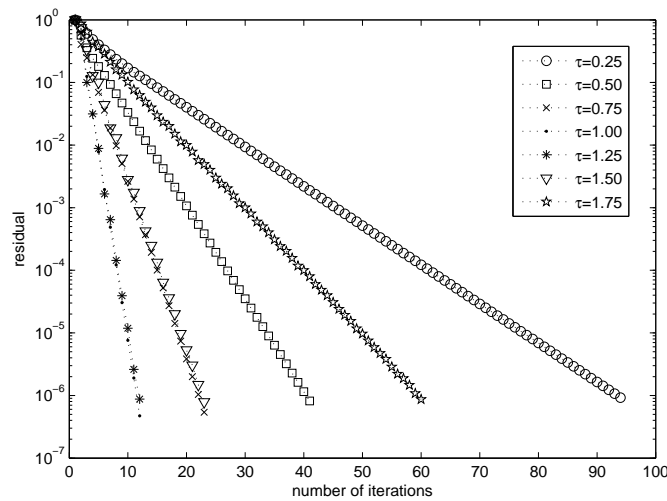


Fig. 5.4. Convergence history for different τ in the case of 20000 time steps.

curve of number of iterations versa parameter τ in the case of 20000 time steps. Obviously, the optimal parameter is around 1. In Fig. 5.4, the convergence history of the SSORWR method for different τ is plotted in the case of 20000 time steps. The scale of the residual is logarithmic. We see that the SSORWR method is linearly convergent for all τ 's. We remark that the situation for other cases, i.e., different number of time steps, is similar.

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