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FINITE DIFFERENCE METHODS FOR THE HEAT EQUATION WITH A NONLOCAL BOUNDARY CONDITION*

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Abstract

We consider the numerical solution by finite difference methods of the heat equation in one space dimension, with a nonlocal integral boundary condition, resulting from the truncation to a finite interval of the problem on a semi-infinite interval. We first analyze the forward Euler method, and then the $\theta-$method for $0 < \theta \leq 1$, in both cases in maximum-norm, showing $O(h^2 + k)$ error bounds, where h is the mesh-width and k the time step. We then give an alternative analysis for the case $\theta = 1/2$, the Crank-Nicolson method, using energy arguments, yielding a $O(h^2 + k^{3/2})$ error bound. Special attention is given the approximation of the boundary integral operator. Our results are illustrated by numerical examples.

Mathematics subject classification: 65M06, 65M12, 65M15
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1. Introduction

We are concerned with the numerical solution of the parabolic problem on a semi-infinite interval,

$$u_t = u_{xx} + f(x,t), \qquad \text{for } x \geq 0, \quad t > 0, \tag{1.1a}$$
$$u(0,t) = b(t), \qquad \text{for } t > 0, \tag{1.1b}$$
$$u(x,0) = v(x), \qquad \text{for } x \geq 0, \tag{1.1c}$$
$$u \to 0, \qquad \text{for } x \to +\infty, \tag{1.1d}$$

where $f(x,t)$ and $v(x)$ vanish outside a finite interval in x, which in the sequel we normalize to be $[0,1)$. To be able to use finite difference or finite element methods for this problem, it is useful to truncate it to this finite spatial interval. This necessitates setting up a boundary condition at the right hand endpoint of the interval, $x = 1$, usually referred to as an artificial boundary condition (*abc*). Han and Huang [3] have recently proposed such an *abc* for (1.1)

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resulting in the initial-boundary value problem

$$u_t = u_{xx} + f(x,t), \qquad \text{for } x \in (0,1), \quad t > 0 \tag{1.2a}$$
$$u(0,t) = b(t), \qquad \text{for } t > 0, \tag{1.2b}$$
$$u_x(1,t) + Gu(1,t) = g(t), \qquad \text{for } t > 0, \tag{1.2c}$$
$$u(x,0) = v(x), \qquad \text{for } x \in (0,1), \tag{1.2d}$$

with $g(t) = 0$, where Gu may be thought of as a fractional derivative of order $\frac{1}{2}$ of u, cf. [8], or

$$Gu(t) = Ju_t(t), \quad \text{where } Jw(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{w(s)}{\sqrt{t-s}} ds. \tag{1.3}$$

The function $g(t)$ will be included below for the purpose of our analysis.

To derive this *abc* at $x = 1$, we set $b_1(t) = u(1,t)$, with u the solution of (1.1), and note that u also solves

$$u_t = u_{xx}, \qquad \text{for } x \geq 1, \quad t > 0,$$
$$u(1,t) = b_1(t), \qquad \text{for } t > 0, \quad \text{and} \quad u(x,0) = 0, \quad \text{for } x \geq 1.$$

Using Laplace transformation one shows that the solution of this problem is

$$u(x,t) = \frac{x-1}{2\sqrt{\pi}} \int_0^t (t-s)^{-3/2} b_1(s) e^{-(x-1)^2/(4(t-s))}\, ds, \quad \text{for } x > 1,\ t > 0.$$

From this one finds, after some calculation, that

$$u_x(1,t) = -\frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2} b_1'(s)\, ds = -Ju_t(1,t), \quad \text{for } t > 0,$$

and hence that the boundary condition at $x = 1$ in (1.2) holds. Although [3] does not conatin any error analysis, the authors demonstrated the effectiveness of this *abc* by numerical computation. Recently Wu and Sun [7] have analyzed this *abc* for a slightly more complicated difference scheme than the Crank-Nicolson one, and Zheng [8] employs the same condition for the time discretized heat equation using the $\mathcal{Z}$ transform. For a technique that does not truncate the domain, see Li and Greengard [4]. Tsynkov [6] contains a survey of numerical solution on infinite domains.

Our purpose here is to analyze the solution of the truncated problem (1.2) by finite differences, using the θ-method, for $0 \leq \theta \leq 1$. For $\theta = 0$ this reduces to the explicit forward Euler method, and for $\theta > 0$ the method is implicit, with the backward Euler method corresponding to $\theta = 1$, and the Crank-Nicolson method to $\theta = \frac{1}{2}$.

We use the spatial grid $x_m = mh, m = 0, 1, \dots M+1$, with $h = 1/M'$, where M is a positive integer and $M' = M + \frac{1}{2}$, thus also using the grid point $x_{M+1} = 1 + \frac{1}{2}h$ to the right of the right hand boundary, but with no gridpoint at $x = 1$. The step size in time is denoted by k, with the corresponding time levels $t_n = nk$. We denote by U_m^n the difference approximation of $u(x_m, t_n)$ and introduce the forward and backward difference quotients in space and time by

$$\partial_x U_m^n = \frac{U_{m+1}^n - U_m^n}{h}, \qquad \bar\partial_x U_m^n = \frac{U_m^n - U_{m-1}^n}{h},$$
$$\partial_t U_m^n = \frac{U_m^{n+1} - U_m^n}{k}, \qquad \bar\partial_t U_m^n = \frac{U_m^n - U_m^{n-1}}{k}.$$