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TWO-STEP MODULUS-BASED SYNCHRONOUS MULTISPLITTING ITERATION METHODS FOR LINEAR COMPLEMENTARITY PROBLEMS*

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Abstract

To reduce the communication among processors and improve the computing time for solving linear complementarity problems, we present a two-step modulus-based synchronous multisplitting iteration method and the corresponding symmetric modulus-based multisplitting relaxation methods. The convergence theorems are established when the system matrix is an H_+-matrix, which improve the existing convergence theory. Numerical results show that the symmetric modulus-based multisplitting relaxation methods are effective in actual implementation.

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Key words: Linear complementarity problem, Modulus-based method, Matrix multisplitting, Convergence.

1. Introduction

Given a real matrix $A \in \mathbb{R}^{n\times n}$ and a real vector $q \in \mathbb{R}^n$, the linear complementarity problem abbreviated as LCP(q, A) is to find a pair of real vectors $r, z \in \mathbb{R}^n$ such that

$$r := Az + q \geq 0, \quad z \geq 0 \quad \text{and} \quad z^T(Az + q) = 0.$$

The linear complementarity problem has extensive applications in the field of economy and engineering; see [11,14]. The modulus method is one of the classic iteration methods for solving linear complementarity problems; see, e.g., [13,21,24]. More recently, Hadjidimos and Tzoumas presented the extrapolated modulus algorithms in [17,18], and Bai presented the modulus-based matrix splitting iteration method in [3]. These two new methods are very effective and practical in numerical computation.

For large sparse linear complementarity problems arising in the engineering applications, the multisplitting iterative methods are powerful tools to enlarge the scale of problem and speed up the computation; see, e.g., [1, 2, 4, 5, 7, 12, 22]. Recently, by an equivalent reformulation of the linear complementarity problem into a system of fixed-point equations, Bai and Zhang have constructed the *modulus-based synchronous multisplitting* (**MSM**) iteration methods in [7], which are suitable to be implemented parallelly on multiprocessor systems. As the communication among processors is much more time-consuming than the computation, we intend to reduce

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the communication by making full use of the previous iteration and communication. To this end, we present the two-step modulus-based synchronous multisplitting iteration methods as well as their relaxed variants in this paper, which consist of two sweeps at each iteration step. We remark that these two-step methods are different from the *two-stage* methods presented in [8, 27], which are inner/outer iteration methods aimed to solve the outer iteration efficiently.

The remaining part of this paper is organized as follows: In Section 2, we introduce some notations and briefly review the MSM iteration methods. In Section 3, we propose the two-step modulus-based synchronous multisplitting iteration methods as well as their relaxed variants. In Section 4, we prove their convergence when the system matrix is an H_+-matrix. Numerical results are given in Section 5. Finally, we make a conclusion in Section 6.

2. Notations and Preliminaries

For $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{m\times n}$, we write $A \geq B$ $(A > B)$ if $a_{ij} \geq b_{ij}$ $(a_{ij} > b_{ij})$ hold for all $1 \leq i \leq m$ and $1 \leq j \leq n$. If O is the null matrix and $A \geq O$ $(A > O)$, we say that A is a nonnegative (positive) matrix. $|A|$ and A^T denote the absolute value and the transpose of the matrix A, respectively.

For a square matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, we denote its spectral radius and diagonal part by $\rho(A)$ and $\operatorname{diag}(A)$, respectively. Its comparison matrix $\langle A\rangle = (\langle a_{ij}\rangle)$ is defined by $\langle a_{ij}\rangle = |a_{ij}|$ if $i = j$ and $\langle a_{ij}\rangle = -|a_{ij}|$ if $i \neq j$. It is called an M-matrix if its off-diagonal entries are all non-positive and $A^{-1} \geq O$, an H-matrix if its comparison matrix $\langle A\rangle$ is an M-matrix, and an H_+-matrix if it is an H-matrix with positive diagonal entries [2, 9, 25]. Note that if A is an H_+-matrix, then $\rho(D^{-1}|B|) < 1$, where $D = \operatorname{diag}(A)$ and $B = D - A$; see [9]. In this paper, we focus on the case that A is an H_+-matrix, which is a sufficient condition for LCP(q, A) to possess a unique solution for any q.

If A is an M-matrix and Λ is a positive diagonal matrix, then $A \leq B \leq \Lambda$ implies that B is an M-matrix. If A is an H-matrix, then A is nonsingular and $|A^{-1}| \leq \langle A\rangle^{-1}$; see, e.g., [9, 15]. The splitting $A = M - N$ is called an H-compatible splitting if it satisfies $\langle A\rangle = \langle M\rangle - |N|$; see, e.g., [16].

Lemma 2.1. ([19,20]). *Let $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ be a strictly diagonally dominant matrix. Then*

$$\left\|M^{-1}N\right\|_\infty \leq \max_{1\leq i\leq n} \frac{\sum\limits_{j=1}^{n} |n_{ij}|}{|m_{ii}| - \sum\limits_{j\neq i} |m_{ij}|}$$

holds for any matrix $N = (n_{ij}) \in \mathbb{R}^{n\times n}$.

Lemma 2.2. ([3]). *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, Ω be a positive diagonal matrix, and γ be a positive constant. For the LCP (q, A), the following statements hold true:*

(i) if (z, r) is a solution of the LCP(q, A), then $x = \frac{1}{2}\gamma(z - \Omega^{-1}r)$, with $|x| = \frac{1}{2}\gamma(z + \Omega^{-1}r)$, satisfies the implicit fixed-point equation

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q; \tag{2.1}$$

(ii) if x satisfies the implicit fixed-point Eq. (2.1), *then*

$$z = \gamma^{-1}(|x| + x) \quad \textit{and} \quad r = \gamma^{-1}\Omega(|x| - x)$$

is a solution of the LCP(q, A).

To precisely describe the MSM iteration method, we first state the concept of matrix multisplitting. Let ℓ be a given positive integer with $\ell \leq n$, $A = M_k - N_k$ $(k = 1, \ldots, \ell)$ be splittings of the system matrix $A \in \mathbb{R}^{n \times n}$, and $E_k \in \mathbb{R}^{n \times n}$ $(k = 1, \ldots, \ell)$ be nonnegative diagonal matrices satisfying $\sum_{k=1}^{\ell} E_k = I$ (the identity matrix). Then the collection of triples (M_k, N_k, E_k) $(k = 1, \ldots, \ell)$ is called a multisplitting of the matrix A, and the matrices E_k $(k = 1, \ldots, \ell)$ are called weighting matrices; see [6,23]. Assume that (M_k, N_k, E_k) $(k = 1, \ldots, \ell)$ is a multisplitting of the system matrix $A \in \mathbb{R}^{n \times n}$, γ is a positive constant and Ω is a positive diagonal matrix. Then the MSM iteration method established in [7] can be described as follows.

Method 2.1. (The MSM Iteration Method for LCP(q, A))

Step 1. *Choose an initial vector $x^{(0)} \in \mathbb{R}^n$, and set $m := 0$;*

Step 2. *For $k = 1, \ldots, \ell$, we solve the linear subsystem*

$$(\Omega + M_k)x^{(m+1,k)} = N_k x^{(m)} + (\Omega - A)|x^{(m)}| - \gamma q,$$

on the k-th processor, and obtain the solution $x^{(m+1,k)}$;

Step 3. *By combining the local updates of ℓ processors together, we get*

$$x^{(m+1)} = \sum_{k=1}^{\ell} E_k x^{(m+1,k)} \quad \textit{and} \quad z^{(m+1)} = \frac{1}{\gamma}\left(|x^{(m+1)}| + x^{(m+1)}\right);$$

Step 4. *If $z^{(m+1)}$ satisfies a prescribed stopping rule, then terminate. Otherwise, set $m := m + 1$ and return to Step 2.*

3. The Two-Step Modulus-Based Multisplitting Iteration Method

From the numerical results in [26], we know that for some linear complementarity problems the two-step modulus-based matrix splitting iteration method is effective to decrease the number of iteration steps. Thus, for the MSM iteration method we take two sweeps at each iteration step to reduce the communication among the processors, which may improve the computing time for solving linear complementarity problems. We shall call this iteration method the *two-step modulus-based synchronous multisplitting* (**TMSM**) iteration method. Assume that (M'_k, N'_k, E_k) and (M''_k, N''_k, E_k) $(k = 1, \ldots, \ell)$ are two multisplittings of the system matrix $A \in \mathbb{R}^{n \times n}$, the TMSM iteration method is as follows:

Method 3.1. (The TMSM Iteration Method for LCP(q, A))

Step 1. *Choose an initial vector $x^{(0)} \in \mathbb{R}^n$, and set $m := 0$;*

Step 2. *For $k = 1, \ldots, \ell$, we solve the linear subsystem*

$$\begin{cases} (\Omega + M_k')x^{(m+\frac{1}{2},k)} = N_k' x^{(m)} + (\Omega - A)|x^{(m)}| - \gamma q, \\ (\Omega + M_k'')x^{(m+1,k)} = N_k'' x^{(m+\frac{1}{2},k)} + (\Omega - A)|x^{(m+\frac{1}{2},k)}| - \gamma q, \end{cases} \tag{3.1}$$

on the k-th processor, and obtain the solution $x^{(m+1,k)}$;

Step 3. *By combining the local updates of ℓ processors together, we get*

$$x^{(m+1)} = \sum_{k=1}^{\ell} E_k x^{(m+1,k)} \quad \text{and} \quad z^{(m+1)} = \frac{1}{\gamma}\left(|x^{(m+1)}| + x^{(m+1)}\right);$$

Step 4. *If $z^{(m+1)}$ satisfies a prescribed stopping rule, then terminate. Otherwise, set $m := m + 1$ and return to Step 2.*

Similar to the MSM iteration method, we can choose the weighting matrices E_k $(k = 1, \ldots, \ell)$ suitably such that the tasks distributed on the ℓ processors are well balanced. We remark that when $\ell = 1$, the TMSM iteration method naturally reduces to the two-step modulus-based matrix splitting iteration method in [26].

To make the TMSM iteration method more convenient in concrete applications, we consider the usual symmetric relaxation methods. To this end, we let $D = \mathrm{diag}(A)$, L_k' be a strictly lower-triangular matrix and $U_k' = D - L_k' - A$, and U_k'' be a strictly supper-triangular matrix and $L_k'' = D - U_k'' - A$. Here, we call $(D - L_k', U_k', E_k)$ and $(D - U_k'', L_k'', E_k)$ $(k = 1, \ldots, \ell)$ are the triangular multisplittings of the matrix A. Note that U_k' and L_k'' are zero-diagonal matrices. By taking

$$\begin{aligned} M_k' &= \frac{1}{\alpha}(D - \beta L_k'), \quad N_k' = \frac{1}{\alpha}[(1-\alpha)D + (\alpha - \beta)L_k' + \alpha U_k'], \\ M_k'' &= \frac{1}{\alpha}(D - \beta U_k''), \quad N_k'' = \frac{1}{\alpha}[(1-\alpha)D + (\alpha - \beta)U_k'' + \alpha L_k''], \end{aligned}$$

in Method 3.1, where α and β are prescribed relaxation parameters, we can obtain a *symmetric modulus-based synchronous multisplitting accelerated overrelaxation* (**SMSMAOR**) iteration method, in which the local update $x^{(m+1,k)}$ is obtained by solving the triangular linear systems

$$\begin{cases} (\alpha\Omega + D - \beta L_k')x^{(m+\frac{1}{2},k)} \\ \qquad = [(1-\alpha)D + (\alpha - \beta)L_k' + \alpha U_k']x^{(m)} + \alpha[(\Omega - A)|x^{(m)}| - \gamma q], \\ (\alpha\Omega + D - \beta U_k'')x^{(m+1,k)} \\ \qquad = [(1-\alpha)D + (\alpha - \beta)U_k'' + \alpha L_k'']x^{(m+\frac{1}{2},k)} + \alpha[(\Omega - A)|x^{(m+\frac{1}{2},k)}| - \gamma q]. \end{cases}$$

In the SMSMAOR iteration method, a forward sweep is followed by a backward sweep at each iteration step.

If we choose the parameter pair (α, β) to be (α, α) and $(1,1)$, respectively, the SMSMAOR method reduces to the so-called SMSMSOR and SMSMGS methods, correspondingly. These relaxation methods are quite practical and efficient for solving large sparse linear complementarity problems on the high-speed multiprocessor systems.

4. Convergence Theorems

In this section, we firstly improve the convergence theorems of the two-step modulus-based matrix splitting iteration method in [26]. Then, we prove the convergence of the TMSM and SMSMAOR iteration methods.

Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix and $A = M - N$ be an H-compatible splitting. Denote by

$$\mathcal{L}(M,N,\Omega) = (\Omega + \langle M\rangle)^{-1}(|\Omega - A| + |N|), \tag{4.1}$$

we can prove the following conclusion.

Theorem 4.1. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an H_+-matrix with $D = \mathrm{diag}(A)$ and $B = D - A$, $A = M - N$ be an H-compatible splitting with $M = (m_{ij}) \in \mathbb{R}^{n\times n}$. Assume that $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n) \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix satisfying $\Omega \geq D$. Then*

$$\rho(\mathcal{L}(M,N,\Omega)) \leq 1 - 2\Big(1 - \rho(D^{-1}|B|)\Big) \min_{1\leq i\leq n} \frac{a_{ii}}{\omega_i + m_{ii}} < 1,$$

where $D^{-1}B$ is the Jacobi matrix associated with A.

Proof. We construct an irreducible matrix $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{n\times n}$ as

$$\tilde{a}_{ij} = \begin{cases} a_{ij}, & a_{ij} \neq 0, \\ \varepsilon, & a_{ij} = 0, \end{cases} \qquad i,j = 1,\ldots,n.$$

Obviously, $\mathrm{diag}(\tilde{A}) = D$. Let $\tilde{A} = \tilde{M} - \tilde{N}$ be the corresponding H-compatible splitting which satisfies: if $a_{ij} \neq 0$, then $\tilde{m}_{ij} = m_{ij}$, $\tilde{n}_{ij} = n_{ij}$; if $a_{ij} = 0$, then $\tilde{m}_{ij} = \varepsilon$, $\tilde{n}_{ij} = 0$. Similar to the proof of Lemma 4.1 in [26], for sufficiently small $\varepsilon > 0$, it holds that

$$\begin{aligned} &0 < \rho(D^{-1}|\tilde{B}|) < 1, \\ &\sum_{j\neq i} |\tilde{a}_{ij}| v_j = \rho(D^{-1}|\tilde{B}|) a_{ii} v_i, \quad 1 \leq i \leq n, \end{aligned}$$

where $\tilde{B} = D - \tilde{A}$, $v = (v_1, \ldots, v_n)^T > 0$ is a positive vector.

Let $V = \mathrm{diag}(v_1, \ldots, v_n)$. We can prove that $\Omega V + \langle\tilde{M}\rangle V$ is a strictly diagonally dominant matrix. Then it follows from Lemma 2.1 that

$$\begin{aligned} \big\|V^{-1}\mathcal{L}(\tilde{M},\tilde{N},\Omega)V\big\|_\infty &= \big\|(\Omega V + \langle\tilde{M}\rangle V)^{-1}(|\Omega - \tilde{A}|V + |\tilde{N}|V)\big\|_\infty \\ &\leq \max_{1\leq i\leq n} \frac{(\omega_i - a_{ii})v_i + n_{ii}v_i + \sum_{j\neq i} |\tilde{a}_{ij}|v_j + \sum_{j\neq i} |\tilde{n}_{ij}|v_j}{(\omega_i + m_{ii})v_i - \sum_{j\neq i} |\tilde{m}_{ij}|v_j} \\ &= \max_{1\leq i\leq n} \frac{\big[\omega_i + m_{ii} - 2(1 - \rho(D^{-1}|\tilde{B}|))a_{ii}\big]v_i - \sum_{j\neq i} |\tilde{m}_{ij}|v_j}{(\omega_i + m_{ii})v_i - \sum_{j\neq i} |\tilde{m}_{ij}|v_j} \\ &\leq \max_{1\leq i\leq n} \frac{\omega_i + m_{ii} - 2(1 - \rho(D^{-1}|\tilde{B}|))a_{ii}}{\omega_i + m_{ii}}. \end{aligned}$$

Here, we make use of the inequality $\frac{b-c}{a-c} \leq \frac{b}{a}$, where the positive constants a, b, c satisfy $a-c > 0$ and $a \geq b$. Thus, we obtain

$$\begin{aligned} &\rho(\mathcal{L}(M,N,\Omega)) = \lim_{\varepsilon\to 0} \rho(\mathcal{L}(\tilde{M},\tilde{N},\Omega)) \\ =&\lim_{\varepsilon\to 0} \rho(V^{-1}\mathcal{L}(\tilde{M},\tilde{N},\Omega)V) \leq \lim_{\varepsilon\to 0} \left\|V^{-1}\mathcal{L}(\tilde{M},\tilde{N},\Omega)V\right\|_\infty \\ \leq& \lim_{\varepsilon\to 0} \max_{1\leq i\leq n} \frac{\omega_i + m_{ii} - 2(1-\rho(D^{-1}|\tilde{B}|))a_{ii}}{\omega_i + m_{ii}} \\ =& 1 - 2(1-\rho(D^{-1}|B|)) \min_{1\leq i\leq n} \frac{a_{ii}}{\omega_i + m_{ii}} < 1, \end{aligned}$$

since $\rho(D^{-1}|B|) < 1$. $\square$

We remark that Theorem 4.1 requires $\Omega \geq D$, while Lemma 4.1 in [26] requires $\Omega \geq \mathrm{diag}(M)$. Since $\mathrm{diag}(M) \geq D$, we see that the former is an improvement of the latter. In (4.1), let

$$M = \frac{1}{\alpha}(D - \beta L), \quad N = \frac{1}{\alpha}((1-\alpha)D + (\alpha-\beta)L + \alpha U),$$

where $D = \mathrm{diag}(A)$, L is the strictly lower-triangular matrix, and U is the zero-diagonal matrix satisfying $A = D - L - U$. Denote by

$$\begin{aligned} &\mathcal{L}_{\mathrm{MAOR}}(M,N,\Omega) \\ =&\Big(\Omega + \frac{1}{\alpha}D - \frac{\beta}{\alpha}|L|\Big)^{-1} \cdot \Big(|\Omega - A| + \frac{1}{\alpha}|(1-\alpha)D + (\alpha-\beta)L + \alpha U|\Big), \end{aligned}$$

we can prove the following conclusion.

Theorem 4.2. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an H_+-matrix with $D = \mathrm{diag}(A)$ and $B = D - A$, L is the strictly lower-triangular matrix, and U is the zero-diagonal matrix satisfying $A = D - L - U$ and $\langle A\rangle = D - |L| - |U|$. Assume that $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n) \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix satisfying $\Omega \geq D$. Then*

$$\rho(\mathcal{L}_{\mathrm{MAOR}}(M,N,\Omega)) \leq \max_{1\leq i\leq n} \frac{\omega_i + \frac{|1-\alpha|-\alpha}{\alpha}a_{ii} + 2\rho(D^{-1}|B|)a_{ii}}{\omega_i + \frac{1}{\alpha}a_{ii}} < 1,$$

provided that $0 < \beta \leq \alpha < \frac{1}{\rho(D^{-1}|B|)}$.

Proof. We construct an irreducible matrix $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{n\times n}$ as

$$\tilde{a}_{ij} = \begin{cases} a_{ij}, & a_{ij} \neq 0, \\ \varepsilon, & a_{ij} = 0, \end{cases} \qquad i,j = 1,\ldots,n.$$

Let $\tilde{A} = D - \tilde{L} - \tilde{U} = D - \tilde{B}$, which satisfies: if $a_{ij} \neq 0$, then $\tilde{l}_{ij} = l_{ij}$, $\tilde{u}_{ij} = u_{ij}$; if $a_{ij} = 0$, then $\tilde{l}_{ij} = \varepsilon$, $\tilde{u}_{ij} = 0$. Consequently, $\langle\tilde{A}\rangle = D - |\tilde{L}| - |\tilde{U}|$. It follows from $0 < \alpha < \frac{1}{\rho(D^{-1}|B|)}$ that

$$\frac{|1-\alpha| - \alpha}{\alpha} + 2\rho(D^{-1}|B|) < \frac{1}{\alpha}.$$

For sufficiently small $\varepsilon > 0$, it holds that

$$\begin{aligned} &0 < \rho(D^{-1}|\tilde{B}|) < 1, \quad \frac{|1-\alpha| - \alpha}{\alpha} + 2\rho(D^{-1}|\tilde{B}|) < \frac{1}{\alpha}, \\ &\sum_{j\neq i} |\tilde{a}_{ij}|v_j = \rho(D^{-1}|\tilde{B}|)a_{ii}v_i, \quad 1 \leq i \leq n, \end{aligned}$$

where $v=(v_1,\ldots,v_n)^T>0$ is a positive vector. Analogously, we take

$$\tilde{M}=\frac{1}{\alpha}(D-\beta\tilde{L}),\quad \tilde{N}=\frac{1}{\alpha}((1-\alpha)D+(\alpha-\beta)\tilde{L}+\alpha\tilde{U}).$$

Let $V=\operatorname{diag}(v_1,\ldots,v_n)$. Then

$$\begin{aligned}&V^{-1}\mathcal{L}_{\mathrm{MAOR}}(\tilde{M},\tilde{N},\Omega)V\\ =&\Big(\Omega V+\frac{1}{\alpha}DV-\frac{\beta}{\alpha}|\tilde{L}|V\Big)^{-1}\cdot\Big(|\Omega-A|V+\frac{1}{\alpha}|(1-\alpha)D+(\alpha-\beta)\tilde{L}+\alpha\tilde{U}|V\Big).\end{aligned}$$

From $\Omega\ge D$ and $0<\beta\le\alpha$, we can prove that $\Omega V+\frac{1}{\alpha}DV-\frac{\beta}{\alpha}|\tilde{L}|V$ is a strictly diagonally dominant matrix. Thus, by Lemma 2.1 we obtain

$$\begin{aligned}&\left\|V^{-1}\mathcal{L}_{\mathrm{MAOR}}(\tilde{M},\tilde{N},\Omega)V\right\|_\infty\\ \le&\max_{1\le i\le n}\frac{\Big(\omega_i+\frac{|1-\alpha|-\alpha}{\alpha}a_{ii}\Big)v_i+2\sum_{j\ne i}|\tilde{a}_{ij}|v_j-\frac{\beta}{\alpha}\sum_{j=1}^{i-1}|\tilde{l}_{ij}|v_j}{\omega_iv_i+\frac{1}{\alpha}a_{ii}v_i-\frac{\beta}{\alpha}\sum_{j=1}^{i-1}|\tilde{l}_{ij}|v_j}\\ =&\max_{1\le i\le n}\frac{\Big(\omega_i+\frac{|1-\alpha|-\alpha}{\alpha}a_{ii}+2\rho(D^{-1}|\tilde{B}|)a_{ii}\Big)v_i-\frac{\beta}{\alpha}\sum_{j=1}^{i-1}|\tilde{l}_{ij}|v_j}{\omega_iv_i+\frac{1}{\alpha}a_{ii}v_i-\frac{\beta}{\alpha}\sum_{j=1}^{i-1}|\tilde{l}_{ij}|v_j}\\ \le&\max_{1\le i\le n}\frac{\omega_i+\frac{|1-\alpha|-\alpha}{\alpha}a_{ii}+2\rho(D^{-1}|\tilde{B}|)a_{ii}}{\omega_i+\frac{1}{\alpha}a_{ii}}.\end{aligned}$$

Here, we make use of the relations $|\tilde{B}|=|\tilde{L}|+|\tilde{U}|$ and $\frac{b-c}{a-c}\le\frac{b}{a}$, where the positive constants a,b,c satisfy $a-c>0$ and $a\ge b$. Thus, we obtain

$$\begin{aligned}\rho(\mathcal{L}_{\mathrm{MAOR}}(M,N,\Omega))&=\lim_{\varepsilon\to0}\rho(\mathcal{L}_{\mathrm{MAOR}}(\tilde{M},\tilde{N},\Omega))\\ &=\lim_{\varepsilon\to0}\rho(V^{-1}\mathcal{L}_{\mathrm{MAOR}}(\tilde{M},\tilde{N},\Omega)V)\\ &\le\lim_{\varepsilon\to0}\left\|V^{-1}\mathcal{L}_{\mathrm{MAOR}}(\tilde{M},\tilde{N},\Omega)V\right\|_\infty\\ &\le\lim_{\varepsilon\to0}\max_{1\le i\le n}\frac{\omega_i+\frac{|1-\alpha|-\alpha}{\alpha}a_{ii}+2\rho(D^{-1}|\tilde{B}|)a_{ii}}{\omega_i+\frac{1}{\alpha}a_{ii}}\\ &=\max_{1\le i\le n}\frac{\omega_i+\frac{|1-\alpha|-\alpha}{\alpha}a_{ii}+2\rho(D^{-1}|B|)a_{ii}}{\omega_i+\frac{1}{\alpha}a_{ii}}<1.\end{aligned}$$

This completes the proof of the theorem. □

Compared with Theorem 4.3 in [26], we note that in Theorem 4.2:

(i) The positive diagonal matrix Ω satisfies $\Omega\ge D$;

(ii) The upper bound of the relaxation parameters is $\frac{1}{\rho(D^{-1}|B|)}$, which is an improvement of $\frac{2}{1+\rho(D^{-1}|B|)}$;

(iii) L is the strictly lower-triangular matrix, U is the zero-diagonal matrix satisfying $A=D-L-U$ and $\langle A\rangle=D-|L|-|U|$.

Theorem 4.3. *Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix with $D = \operatorname{diag}(A)$, (M'_k, N'_k, E_k) and (M''_k, N''_k, E_k) $(k = 1,\dots,\ell)$ be two multisplittings of the matrix A. Assume that $A = M'_k - N'_k$ and $A = M''_k - N''_k$ are H-compatible splittings for $k = 1,\dots,\ell$, and Ω is a positive diagonal matrix satisfying $\Omega \ge D$. Then for any initial vector $x^{(0)} \in \mathbb{R}^n$, the sequence $\{z^{(m)}\}_{m=0}^{\infty} \subset \mathbb{R}^n_+$ generated by the TMSM iteration method converges to the unique solution z_* of the LCP(q,A).*

Proof. Assume that (z_*, r_*) is the solution of the LCP(q,A). It follows from Lemma 2.2 that $x_* = \frac{1}{2}\gamma(z_* - \Omega^{-1}r_*)$ satisfies the iterative format (3.1). Hence, we can get the following error relationship of the TMSM iteration method

$$\begin{cases} (\Omega + M'_k)(x^{(m+\frac{1}{2},k)} - x_*) = N'_k(x^{(m)} - x_*) + (\Omega - A)(|x^{(m)}| - |x_*|), \\ (\Omega + M''_k)(x^{(m+1,k)} - x_*) = N''_k(x^{(m+\frac{1}{2},k)} - x_*) + (\Omega - A)(|x^{(m+\frac{1}{2},k)}| - |x_*|). \end{cases}$$

Combining the analyses of the MSM iteration method in [7] and the two-step modulus-based matrix splitting iteration method in [26], we know that the errors of the TMSM iteration method satisfy

$$|x^{(m+1)} - x_*| \le \sum_{k=1}^{\ell} E_k \mathcal{L}(M''_k, N''_k, \Omega)\mathcal{L}(M'_k, N'_k, \Omega)|x^{(m)} - x_*|, \tag{4.2}$$

where

$$\begin{aligned} \mathcal{L}(M'_k, N'_k, \Omega) &= (\Omega + \langle M'_k\rangle)^{-1}(|\Omega - A| + |N'_k|), \\ \mathcal{L}(M''_k, N''_k, \Omega) &= (\Omega + \langle M''_k\rangle)^{-1}(|\Omega - A| + |N''_k|). \end{aligned}$$

For simplicity, we denote the iterative matrix of (4.2) by

$$\mathcal{L}_{\mathrm{TMSM}} = \sum_{k=1}^{\ell} E_k \mathcal{L}(M''_k, N''_k, \Omega)\mathcal{L}(M'_k, N'_k, \Omega).$$

Similar to Theorem 4.1, we construct the irreducible H_+-matrix $\tilde{A}$ and its corresponding H-compatible splittings

$$\tilde{A} = \tilde{M}'_k - \tilde{N}'_k \ \text{ and } \ \tilde{A} = \tilde{M}''_k - \tilde{N}''_k, \ \text{ for } \ k = 1,\dots,\ell.$$

As $\Omega \ge D$, from the proof of Theorem 4.1 we know that there exists a positive diagonal matrix V such that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \|V^{-1}\mathcal{L}(\tilde{M}'_k, \tilde{N}'_k, \Omega)V\|_\infty < 1, \\ &\lim_{\varepsilon\to 0} \|V^{-1}\mathcal{L}(\tilde{M}''_k, \tilde{N}''_k, \Omega)V\|_\infty < 1 \end{aligned}$$

hold for all $k = 1,\dots,\ell$. Note that V only depends on the matrix $\tilde{A}$. Denote by $E_k = \operatorname{diag}(e_1^{(k)},\dots,e_n^{(k)})$, $k = 1,\dots,\ell$. As E_k $(k = 1,\dots,\ell)$ are nonnegative diagonal matrices and satisfy $\sum_{k=1}^{\ell} E_k = I$, we obtain

$$
\begin{aligned}
\rho(\mathcal{L}_{\mathrm{TMSM}}) &= \rho\Big(\sum_{k=1}^{\ell} E_k \mathcal{L}(M_k'', N_k'', \Omega)\mathcal{L}(M_k', N_k', \Omega)\Big) \\
&= \lim_{\varepsilon\to 0} \rho\Big(\sum_{k=1}^{\ell} E_k \mathcal{L}(\tilde{M}_k'', \tilde{N}_k'', \Omega)\mathcal{L}(\tilde{M}_k', \tilde{N}_k', \Omega)\Big) \\
&= \lim_{\varepsilon\to 0} \rho\Big(\sum_{k=1}^{\ell} E_k V^{-1} \mathcal{L}(\tilde{M}_k'', \tilde{N}_k'', \Omega)\mathcal{L}(\tilde{M}_k', \tilde{N}_k', \Omega)V\Big) \\
&\le \lim_{\varepsilon\to 0} \Big\| \sum_{k=1}^{\ell} E_k V^{-1} \mathcal{L}(\tilde{M}_k'', \tilde{N}_k'', \Omega)\mathcal{L}(\tilde{M}_k', \tilde{N}_k', \Omega)V \Big\|_\infty \\
&\le \lim_{\varepsilon\to 0} \max_{1\le i\le n} \sum_{k=1}^{\ell} e_i^{(k)} \| V^{-1} \mathcal{L}(\tilde{M}_k'', \tilde{N}_k'', \Omega)\mathcal{L}(\tilde{M}_k', \tilde{N}_k', \Omega)V \|_\infty \\
&\le \max_{1\le i\le n} \sum_{k=1}^{\ell} e_i^{(k)} \Big(\lim_{\varepsilon\to 0} \| V^{-1} \mathcal{L}(\tilde{M}_k'', \tilde{N}_k'', \Omega)V \|_\infty \Big)\Big(\lim_{\varepsilon\to 0} \| V^{-1} \mathcal{L}(\tilde{M}_k', \tilde{N}_k', \Omega)V \|_\infty \Big) \\
&< 1.
\end{aligned}
$$

This completes the proof. □

Theorem 4.4. *Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix with $D = \mathrm{diag}(A)$ and $B = D - A$, $(D - L_k', U_k', E_k)$ and $(D - U_k'', L_k'', E_k)$ $(k = 1, \ldots, \ell)$ be two triangular multisplittings, where L_k' and U_k'' are the strictly lower-triangular and the strictly upper-triangular matrices, respectively. Assume that for $k = 1, \ldots, \ell$, $A = D - L_k' - U_k'$ satisfy $\langle A\rangle = D - |L_k'| - |U_k'|$, $A = D - L_k'' - U_k''$ satisfy $\langle A\rangle = D - |L_k''| - |U_k''|$, and the positive diagonal matrix $\Omega \ge D$. Then, for any initial vector, the following statements hold true:*

(i) *the SMSMGS iteration method is convergent;*

(ii) *the SMSMSOR iteration method is convergent for $0 < \alpha < \frac{1}{\rho(D^{-1}|B|)}$;*

(iii) *the SMSMAOR iteration method is convergent for $0 < \beta \le \alpha < \frac{1}{\rho(D^{-1}|B|)}$.*

Proof. We only need to verify the validity of the statement (iii), as the statements (i) and (ii) are its special cases. Now, take

$$
\begin{aligned}
M_k' &= \frac{1}{\alpha}(D - \beta L_k'), \quad N_k' = \frac{1}{\alpha}[(1-\alpha)D + (\alpha - \beta)L_k' + \alpha U_k'], \\
M_k'' &= \frac{1}{\alpha}(D - \beta U_k''), \quad N_k'' = \frac{1}{\alpha}[(1-\alpha)D + (\alpha - \beta)U_k'' + \alpha L_k''].
\end{aligned}
$$

Similar to Theorem 4.2, we construct the irreducible H_+-matrix $\tilde{A}$ and its corresponding splittings $\tilde{A} = D - \tilde{L}_k' - \tilde{U}_k'$ and $\tilde{A} = D - \tilde{U}_k'' - \tilde{L}_k''$ for $k = 1, \ldots, \ell$. As $\Omega \ge D$, from the proof of Theorem 4.2 we know that there exists a positive diagonal matrix V such that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \| V^{-1} \mathcal{L}_{\mathrm{MAOR}}(\tilde{M}_k', \tilde{N}_k', \Omega)V \|_\infty < 1, \\
&\lim_{\varepsilon\to 0} \| V^{-1} \mathcal{L}_{\mathrm{MAOR}}(\tilde{M}_k'', \tilde{N}_k'', \Omega)V \|_\infty < 1
\end{aligned}
$$

hold for all $k = 1, \ldots, \ell$. Note that V only depends on the matrix $\tilde{A}$. The remaining proof is similar to that of Theorem 4.3. □

5. Numerical Results

For comparison, we use the same numerical examples as those in [7] to examine the effectiveness of the TMSM iteration methods in this section. The codes are also written in C and MPICH2, and performed on the same PC clusters. The weighting matrices, the initial vector, the stopping criterion, etc., are the same as those in [7], too. For completeness, we recite them briefly. The weighting matrix E_k is set to be

$$E_k = \mathrm{Diag}(0, \ldots, 0, I_{s_k}, 0, \ldots, 0) \in \mathbb{R}^{n\times n},$$

where the size $s_k = \phi_q + 1$ if $k \le \phi_r$, and $s_k = \phi_q$ otherwise, with ϕ_q and ϕ_r being two nonnegative integers satisfying $n = \phi_q \ell + \phi_r$ and $0 \le \phi_r < \ell$. And accordingly, we take L'_k and U''_k to be the strictly lower-triangular part and the strictly upper-triangular part of the matrix $D - E_k B E_k$, respectively. Moreover, we set $\Omega = D$ and $\gamma = 2$. The initial vector and the stopping criterion are $x^{(0)} = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ and $\|\min\{Az^{(m)} + q, z^{(m)}\}\|_2 < 10^{-5}$, respectively.

Example 5.1. ([7,8]) *The LCP(q, A) is given by*

$$A = \begin{bmatrix} S & -I & -I & & \\ & S & -I & \ddots & \\ & & S & \ddots & -I \\ & & & \ddots & -I \\ & & & & S \end{bmatrix} \in \mathbb{R}^{n\times n}$$

and $q = (-1, 1, -1, 1, \ldots) \in \mathbb{R}^n$*, where* $n = n_o^2$ *with* n_o *being a given positive integer, and* $S = \mathrm{tridiag}(-1, 4, -1) \in \mathbb{R}^{n_o\times n_o}$ *is a tridiagonal matrix.*

Table 5.1: Numerical Results for SMSMGS and SMSMSOR for Example 5.1.

n_o	Method	ℓ	1	2	4	8	16	32	64	128
512	SMSMGS	T_ℓ	6.39	3.91	2.64	1.25	0.42	0.18	0.10	0.08
		$\mathcal{E}_\ell$	1	0.82	0.61	0.64	0.95	1.11	1.00	0.62
		IT_ℓ	313	314	315	316	320	326	339	365
	SMSMSOR	T_ℓ	3.96	2.56	1.65	0.81	0.30	0.14	0.08	0.06
		$\mathcal{E}_\ell$	1	0.77	0.60	0.61	0.83	0.88	0.77	0.52
		IT_ℓ	163	184	190	193	199	206	216	235
1024	SMSMGS	T_ℓ	48.94	30.33	20.64	10.66	5.21	2.43	0.83	0.45
		$\mathcal{E}_\ell$	1	0.81	0.59	0.57	0.59	0.63	0.92	0.85
		IT_ℓ	588	589	590	591	594	601	614	640
	SMSMSOR	T_ℓ	31.34	19.28	12.48	6.47	3.25	1.57	0.61	0.32
		$\mathcal{E}_\ell$	1	0.81	0.63	0.61	0.60	0.62	0.80	0.77
		IT_ℓ	319	340	348	351	359	367	379	399
2048	SMSMGS	T_ℓ	377.59	233.84	159.28	82.45	41.89	21.56	10.25	4.97
		$\mathcal{E}_\ell$	1	0.81	0.59	0.57	0.56	0.55	0.58	0.59
		IT_ℓ	1123	1124	1124	1126	1129	1136	1149	1175
	SMSMSOR	T_ℓ	245.88	148.51	93.93	48.91	25.04	12.76	6.39	3.13
		$\mathcal{E}_\ell$	1	0.83	0.65	0.63	0.61	0.60	0.60	0.61
		IT_ℓ	622	650	651	655	671	679	692	711

Table 5.2: The Experimentally Found Optimal Parameters α for SMSMSOR for Example 5.1.

n_o	ℓ							
	1	2	4	8	16	32	64	128
512	2.9	2.3	2.2	2.2	2.2	2.2	2.3	2.4
1024	2.7	2.4	2.3	2.3	2.3	2.3	2.3	2.4
2048	2.6	2.4	2.4	2.4	2.3	2.3	2.4	2.4

We list in Table 5.1 the elapsed wall time T_ℓ (in seconds), the parallel computing efficiency $\mathcal{E}_\ell = T_1/(\ell T_\ell)$, and the iteration step IT_ℓ of SMSMGS and SMSMSOR methods for solving Example 5.1. The experimentally found optimal parameters α for SMSMSOR method are listed in Table 5.2. From these two tables, we have the following observations and conclusions: (i) SMSMSOR with the experimentally optimal parameter is superior to SMSMGS in computing time as well as iteration step. (ii) Comparing with the numerical results of MSMGS and MSMSOR in Table II in [7], we observe that the iteration steps of SMSMGS and SMSMSOR drop by more than half, and the computing time of SMSMGS and SMSMSOR decreases by about 30% and 40%, respectively. These results show that SMSMGS and SMSMSOR are effective to reduce the communication and improve the computing time. (iii) For Example 5.1, almost all parallel efficiencies exceed 0.5. Some parallel efficiencies even exceed 1.0, and there are significant jumps for $n_o = 512$ from $\ell = 8$ to $\ell = 16$ and for $n_o = 1024$ from $\ell = 32$ to $\ell = 64$. These two phenomena may be caused by the memory systems of the PC clusters.

Example 5.2. ([7, 8]) *The LCP(q, A) is the discretization of the free boundary problem describing flow through a porous dam* [10]*, by the nine-point finite difference scheme with the step length $h = 16/2^\tau$ (τ is a positive integer). The system matrix A is as follows:*

$$A = \mathrm{Tridiag}(T, S, T) = \begin{bmatrix} S & T & & & \\ T & S & \ddots & & \\ & T & \ddots & T & \\ & & \ddots & S & T \\ & & & T & S \end{bmatrix} \in \mathbb{R}^{n\times n}$$

is a block n_t-by-n_t tridiagonal matrix with $S = \mathrm{tridiag}(-4, 20, -4)$ and $T = \mathrm{tridiag}(-1, -4, -1)$ are both n_s-by-n_s tridiagonal matrices, where $n_t = 3 \cdot 2^{\tau-1} - 1$, $n_s = 2 \cdot 2^{\tau-1} - 1$ and $n = n_s n_t$.

We show in Tables 5.3 and 5.4 the numerical results of SMSMGS and SMSMSOR methods for solving Example 5.2. From these two tables, we have the following observations and conclusions: (i) SMSMSOR with the experimentally optimal parameter outperforms SMSMGS in computing time as well as iteration step. (ii) Comparing with the numerical results of MSMGS and MSMSOR in Table IV in [7], we observe that the iteration steps of SMSMGS and SMSMSOR drop by about half and one-third, respectively. For SMSMGS, there is a decrease in computing time by about 20% compared with MSMGS. While for SMSMSOR, its computing time increases by about 15% when $\tau = 6$ and 7, and keeps about the same as that of MSMSOR when $\tau = 8$. (iii) For Example 5.2, almost all parallel efficiencies exceed 0.6. Some parallel efficiencies even exceed 1.0, and there is a significant jump for $\tau = 8$ from $\ell = 4$ to $\ell = 8$. These two phenomena may be caused by the memory systems of the PC clusters. For a fixed τ, the parallel efficiencies fall suddenly for the biggest ℓ, which should be mainly caused by the communication among processors.

Table 5.3: Numerical Results for SMSMGS and SMSMSOR for Example 5.2.

τ	Method	ℓ	1	2	4	8	16	32	64	128
6	SMSMGS	T_ℓ	2.51	1.34	0.68	0.37	0.23	0.18	–	–
		$\mathcal{E}_\ell$	1	0.94	0.92	0.85	0.68	0.44	–	–
		IT_ℓ	4719	4791	4868	5018	5317	5911	–	–
	SMSMSOR	T_ℓ	1.77	0.96	0.50	0.27	0.17	0.14	–	–
		$\mathcal{E}_\ell$	1	0.92	0.89	0.82	0.65	0.40	–	–
		IT_ℓ	3146	3232	3320	3493	3770	4334	–	–
7	SMSMGS	T_ℓ	39.56	20.58	10.45	5.36	2.88	1.69	1.24	–
		$\mathcal{E}_\ell$	1	0.96	0.95	0.92	0.86	0.73	0.50	–
		IT_ℓ	18029	18164	18309	18592	19155	20275	22510	–
	SMSMSOR	T_ℓ	27.82	14.63	7.44	3.87	2.12	1.26	0.95	–
		$\mathcal{E}_\ell$	1	0.95	0.93	0.90	0.82	0.69	0.46	–
		IT_ℓ	12019	12177	12344	12671	13319	14349	16454	–
8	SMSMGS	T_ℓ	754.01	451.82	295.15	107.95	52.53	21.94	12.20	8.57
		$\mathcal{E}_\ell$	1	0.83	0.64	0.87	0.90	1.07	0.97	0.69
		IT_ℓ	68759	69014	69289	69825	70892	73020	77257	85713
	SMSMSOR	T_ℓ	522.83	308.69	200.59	73.62	30.21	16.11	9.14	6.71
		$\mathcal{E}_\ell$	1	0.85	0.65	0.89	1.08	1.01	0.89	0.61
		IT_ℓ	45840	46135	46454	47073	48304	50752	54633	62569

Table 5.4: The Experimentally Found Optimal Parameters α for SMSMSOR for Example 5.2.

τ	Grid	ℓ							
		1	2	4	8	16	32	64	128
6	65 × 97	2.0	2.0	2.0	2.0	2.1	2.4	–	–
7	129 × 193	2.0	2.0	2.0	2.0	2.0	2.1	2.4	–
8	257 × 385	2.0	2.0	2.0	2.0	2.0	2.0	2.1	2.4

6. Concluding Remarks

We end the paper with the following remarks:

(1) Provided that the system matrix is an H_+-matrix, we have proved that the two-step modulus-based multisplitting iteration method and its accordingly symmetric multisplitting relaxation methods are convergent. Moreover, these two convergence theorems improve the existing convergence theory in [26].

(2) Numerical results have illustrated that the symmetric modulus-based multisplitting relaxation methods are effective to reduce the communication among processors and improve the computing time.

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