Journal of Computational Mathematics
Vol.33, No.1, 2015, 33–58.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1407-m4486

ON PRECONDITIONING OF INCOMPRESSIBLE NON-NEWTONIAN FLOW PROBLEMS*

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Abstract

This paper deals with fast and reliable numerical solution methods for the incompressible non-Newtonian Navier-Stokes equations. To handle the nonlinearity of the governing equations, the Picard and Newton methods are used to linearize these coupled partial differential equations. For space discretization we use the finite element method and utilize the two-by-two block structure of the matrices in the arising algebraic systems of equations. The Krylov subspace iterative methods are chosen to solve the linearized discrete systems and the development of computationally and numerically efficient preconditioners for the two-by-two block matrices is the main concern in this paper. In non-Newtonian flows, the viscosity is not constant and its variation is an important factor that effects the performance of some already known preconditioning techniques. In this paper we examine the performance of several preconditioners for variable viscosity applications, and improve them further to be robust with respect to variations in viscosity.

Mathematics subject classification: 65F10, 65F08, 65N30.
Key words: non-Newtonian flows, Navier-Stokes equations, Two-by-two block systems, Krylov subspace methods, Preconditioners.

1. Introduction

Numerical algorithms for incompressible non-Newtonian flows have been intensively studied in the past decades. In non-Newtonian flows the viscosity is not constant and may depend on the velocity, which leads to two nonlinear sources in the governing equations, i.e., the diffusion and convection terms. Due to this, the numerical simulation of the incompressible non-Newtonian flows is more complicated than Newtonian flows, where the viscosity is constant and the only source of nonlinearity in the governing equations is the convection term.

A common approach to solve a nonlinear problem is converting it into a linearized problem, computing the updates of the unknowns by solving the linearized problem and iteratively converging to the true nonlinear solutions. If we consider linearization of both the two nonlinear terms, the variable viscosity Oseen-type problem arises. Ignoring the linearization of the convection term leads to the variable viscosity Stokes-type problem, e.g. [16, 30]. The benefit

* Received January 24, 2014 / Revised version received May 21, 2014 / Accepted July 13, 2014 / Published online December 1, 2014 /

of solving the Stokes-type problem is that efficient solution algorithms are easier to construct compared to the Oseen-type problem. On the other hand, it may take more nonlinear iterations to converge for the Stokes-type problem, typically when the convection is relatively dominant. For each type problem, two well-known linearization methods are used, namely, Picard and Newton iterations. To avoid possible slow convergence rate of Picard iterations and the possibly narrow convergence region of Newton iterations, in this paper a combination of these two iteration methods is utilised. We first carry out some Picard iterations to obtain a reasonably "good" solution, and then use this solution as an initial guess for the Newton iterations. We show that in this way a fast convergence of the nonlinear iterations can be achieved.

For the variable viscosity Oseen- and Stokes-type problems with Picard and Newton iterations, the finite element discretization of the linearized problems results in discrete linear systems of two-by-two block form. Solving the linear systems is the most time-consuming task in the numerical simulations. In this paper, Krylov subspace methods with appropriate preconditioners are chosen to solve the arising linear systems. The kernel of this paper is the construction and the analysis of fast and reliable preconditioning techniques for the variable viscosity Oseen- and Stokes-type problems with both Picard and Newton iterations. As far as the authors know, in earlier works, efficient preconditioners for the variable viscosity Oseen- and Stokes-type problems are only studied for Picard iterations, e.g. [16,18,30].

In the past decades, the most often used preconditioners for incompressible Navier-Stokes equations are originally proposed and analysed for the constant viscosity cases, c.f., the surveys [7,10] and the books [1,14,32]. Due to their algebraic construction, some of these preconditioners can be straightforwardly utilised for the variable viscosity applications. In this paper we choose the augmented Lagrangian preconditioner for the Oseen-type problem (Section 3) and the block lower-triangular and the SIMPLER preconditioners for the Stokes-type problem (Section 4). As variable viscosity is an important factor, a crucial objective for having a fast and reliable preconditioner in this case is the robustness with respect to those variations. In order to fully achieve this objective, we modify the above mentioned preconditioners and also propose some computational improvements. The comparison between the targeted preconditioners and the efficiency of the Oseen- and Stokes-type problems are illustrated in Section 5. Conclusions and future work are outlined in Section 6.

2. Problem Formulation and Linearization

In this paper, we assume that the velocity $\mathbf{u}$ and the pressure p satisfy the following generalized stationary incompressible Navier-Stokes equations:

$$\begin{aligned} -\nabla \cdot (2\nu(D_{\mathrm{II}}(\mathbf{u}), p)\mathbf{D}\mathbf{u}) + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega \end{aligned} \tag{2.1}$$

with boundary conditions given by

$$\begin{aligned} \mathbf{u} &= \mathbf{g}, \quad \text{on } \partial\Omega_D \\ \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - \mathbf{n}p &= 0. \quad \text{on } \partial\Omega_N \end{aligned}$$

Here Ω is a bounded and connected domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, and $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ is its boundary, where $\partial\Omega_D$ and $\partial\Omega_N$ denote the parts of the boundary where Dirichlet and Neumann boundary conditions for $\mathbf{u}$ are imposed, respectively. The terms $\mathbf{f} : \Omega \to \mathbb{R}^d$ and $\mathbf{g}$

are a given force field and Dirichlet boundary data for the velocity. The term $\mathbf{n}$ denotes the outward-pointing unit normal to the boundary.

The term $\mathbf{Du} = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$ denotes the rate-of-deformation tensor and $\nu(\cdot)$ denotes the kinematic viscosity which depends on the second invariant of the rate-of-deformation tensor $D_{\mathrm{II}}(\mathbf{u}) = \frac{1}{2}\mathrm{tr}(\mathbf{D}^2\mathbf{u})$ and the pressure p. The following models are most often used in non-Newtonian fluids:

- [M1] $\nu(D_{\mathrm{II}}(\mathbf{u}), p) = \nu_0 + \tau(D_{\mathrm{II}}(\mathbf{u}))^{\frac{\alpha}{2}}$ describing "power law" non-Newtonian fluids (e.g., [12]);
- [M2] $\nu(D_{\mathrm{II}}(\mathbf{u}), p) = \nu_\infty + (\nu_0 - \nu_\infty)(1 + \beta D_{\mathrm{II}}(\mathbf{u}))^{\frac{\alpha}{2}}$ describing "Carreau law" non-Newtonian fluids;
- [M3] $\nu(D_{\mathrm{II}}(\mathbf{u}), p) = \sqrt{2}\sin\phi p(D_{\mathrm{II}}(\mathbf{u}))^{-\frac{1}{2}}$ describing "Schaeffer law" non-Newtonian fluids (e.g., [33]);
- [M4] non-Newtonian fluids with pressure and shear dependent viscosity (e.g., [21]),

with appropriate parameters ν_0, ν_∞, α, β, τ, ϕ.

In this work we only consider the Bingham model, namely, $\nu(D_{\mathrm{II}}(\mathbf{u})) = \nu_0 + \tau(D_{\mathrm{II}}(\mathbf{u}))^{-\frac{1}{2}}$ (M1 with $\alpha = -1$), which is a special case of "power law" non-Newtonian fluids. Due to the possible singularity of $D_{\mathrm{II}}(\mathbf{u})$, some regularization techniques are required. Here we utilize a widely used regularization method, namely, $\nu(D_{\mathrm{II}}(\mathbf{u})) = \nu_0 + \tau(D_{\mathrm{II}}(\mathbf{u}) + \varepsilon^2)^{-\frac{1}{2}}$ (c.f., [12, 16]). In practice, in order to characterize the Bingham flow well, one needs to choose ε as small as possible. In the modified Bingham model, the variation of the viscosity is represented by the parameter ε, since $\nu_{\min} = \nu_0$ and $\nu_{\max} = O(\varepsilon^{-1})$. Small values of ε leads to a large variation of the viscosity and more difficulties in the numerical simulations. One important aim of this paper is to use a reasonably small ε that describes the Bingham flow well and to balance the computational complexity at the same time.

For the weak formulation of the stationary Navier-Stokes equations (2.1), we define the approximate solution and test spaces for the velocity as

$$\mathbf{H}^1_E = \left\{\mathbf{u} \in \mathcal{H}^1(\Omega)^d | \mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega_D\right\},$$
$$\mathbf{H}^1_{E_0} = \left\{\mathbf{v} \in \mathcal{H}^1(\Omega)^d | \mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega_D\right\},$$
$$\mathcal{H}^1(\Omega)^d = \left\{u_i : \Omega \to \mathbb{R}^d \mid u_i, \frac{\partial u_i}{\partial x_j} \in L_2(\Omega), i, j = 1, \cdots, d\right\},$$

and for the pressure as

$$L_2(\Omega) = \left\{p : \Omega \to \mathbb{R} \mid \int_\Omega p^2 < \infty\right\}.$$

Then the weak formulation reads as follows: Find $\mathbf{u} \in \mathbf{H}^1_E$ and $\mathbf{p} \in L_2(\Omega)$ such that

$$\int_\Omega 2\nu(D_{\mathrm{II}}(\mathbf{u}))\mathbf{Du} : \mathbf{Dv}d\Omega + \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\mathbf{v}d\Omega - \int_\Omega p\nabla\cdot\mathbf{v}d\Omega = \int_\Omega \mathbf{fv}d\Omega, \tag{2.2a}$$
$$\int_\Omega q\nabla\cdot\mathbf{u}d\Omega = 0, \tag{2.2b}$$

for all $\mathbf{v} \in \mathbf{H}^1_{E_0}$ and all $q \in L_2(\Omega)$. The pressure is uniquely defined only up to a constant term. To make it unique, one usually imposes the additional constraint $\int_\Omega p \; d\Omega = 0$. We also assume that the discretization is done using a stable pair of FEM spaces, satisfying the Ladyzhenskaya-Babuška-Brezzi (LBB) condition, cf., e.g., [14].

Since the viscosity function $\nu(D_{\mathrm{II}}(\mathbf{u}))$ also depends on the velocity $\mathbf{u}$, two terms in (2.1) exhibit a nonlinear behavior: $\nabla \cdot (2\nu(D_{\mathrm{II}}(\mathbf{u}))\mathbf{D}\mathbf{u})$ and $(\mathbf{u} \cdot \nabla \mathbf{u})$. As already mentioned, the nonlinearity of the considered problem is handled by some linearization methods. The two well-known and most often used methods are the Newton and Picard methods [14], briefly introduced below.

Let $(\mathbf{u}_0, p_0)$ be an initial guess and let $(\mathbf{u}_k, p_k)$ be the approximate solution at the kth nonlinear iteration. Then we update the velocity and the pressure on the $(k+1)$ iteration as $\mathbf{u}_{k+1} = \mathbf{u}_k + \delta\mathbf{u}_k$, $p_{k+1} = p_k + \delta p_k$ for $k = 0, 1, \cdots$ until convergence, where $\delta\mathbf{u}_k \in \mathbf{H}^1_{E_0}$ and $\delta p_k \in L_2(\Omega)$ (provided $\mathbf{u}_k \in \mathbf{H}^1_E$ and $p_k \in L_2(\Omega)$). Substituting $\mathbf{u}_{k+1}$ and p_{k+1} into the weak formulation (2.2), the correction $(\delta\mathbf{u}_k, \delta p_k)$ should satisfy the following problem:
Find $\delta\mathbf{u}_k \in \mathbf{H}^1_{E_0}$ and $\delta p_k \in L_2(\Omega)$ such that

$$\int_\Omega 2\nu(D_{\mathrm{II}}(\mathbf{u}_k))\mathbf{D}\delta\mathbf{u}_k : \mathbf{D}\mathbf{v} d\Omega + \int_\Omega 2\nu'(D_{\mathrm{II}}(\mathbf{u}_k))[\mathbf{D}\mathbf{u}_k : \mathbf{D}\delta\mathbf{u}_k][\mathbf{D}\mathbf{u}_k : \mathbf{D}\mathbf{v}] d\Omega$$
$$+ \int_\Omega (\mathbf{u}_k \cdot \nabla\delta\mathbf{u}_k) \cdot \mathbf{v} d\Omega + \int_\Omega (\delta\mathbf{u}_k \cdot \nabla\mathbf{u}_k) \cdot \mathbf{v} d\Omega - \int_\Omega \delta p_k \; (\nabla \cdot \mathbf{v}) d\Omega = R_k \qquad (2.3a)$$
$$\int_\Omega q \; (\nabla \cdot \delta\mathbf{u}_k) d\Omega = P_k, \qquad (2.3b)$$

for all $\mathbf{v} \in \mathbf{H}^1_{E_0}$ and $q \in L_2(\Omega)$. The residual terms are obtained as

$$R_k = \int_\Omega \mathbf{f} \cdot \mathbf{v} d\Omega - \int_\Omega 2\nu(D_{\mathrm{II}}(\mathbf{u}_k))\mathbf{D}\mathbf{u}_k : \mathbf{D}\mathbf{v} d\Omega - \int_\Omega (\mathbf{u}_k \cdot \nabla\mathbf{u}_k) \cdot \mathbf{v} d\Omega + \int_\Omega p_k \nabla \cdot \mathbf{v} d\Omega \qquad (2.4a)$$
$$P_k = -\int_\Omega q \; (\nabla \cdot \mathbf{u}_k) d\Omega. \qquad (2.4b)$$

This procedure is refereed to as the Newton linearization method. More details on the Newton method can be found, for example, in [14, 19]. In the regularized Bingham model, i.e., $\nu(D_{\mathrm{II}}(\mathbf{u})) = \nu_0 + \tau(D_{\mathrm{II}}(\mathbf{u}) + \varepsilon^2)^{-\frac{1}{2}}$, the derivative $\nu'(D_{\mathrm{II}}(\mathbf{u}))$ in terms of $D_{\mathrm{II}}(\mathbf{u})$ is $\nu'(D_{\mathrm{II}}(\mathbf{u})) = -\frac{1}{2}\tau(D_{\mathrm{II}}(\mathbf{u}) + \varepsilon^2)^{-\frac{3}{2}}$.

Picard linearization is obtained in a similar way as for the Newton method, except that the terms, i.e., $\int_\Omega (\delta\mathbf{u}_k \cdot \nabla\mathbf{u}_k) \cdot \mathbf{v} d\Omega$ and $\int_\Omega 2\nu'(D_{\mathrm{II}}(\mathbf{u}_k))[\mathbf{D}\mathbf{u}_k : \mathbf{D}\delta\mathbf{u}_k][\mathbf{D}\mathbf{u}_k : \mathbf{D}\mathbf{v}] d\Omega$ are dropped. Thus, the linear problem in Picard method reads as follows:
Find $\delta\mathbf{u}_k \in \mathbf{H}^1_{E_0}$ and $\delta p_k \in L_2(\Omega)$ such that

$$\int_\Omega 2\nu(D_{\mathrm{II}}(\mathbf{u}_k))\mathbf{D}\delta\mathbf{u}_k : \mathbf{D}\mathbf{v} d\Omega + \int_\Omega (\mathbf{u}_k \cdot \nabla\delta\mathbf{u}_k) \cdot \mathbf{v} d\Omega - \int_\Omega \delta p_k \; (\nabla \cdot \mathbf{v}) d\Omega = R_k \qquad (2.5a)$$
$$\int_\Omega q \; (\nabla \cdot \delta\mathbf{u}_k) d\Omega = P_k, \qquad (2.5b)$$

for all $\mathbf{v} \in \mathbf{H}^1_{E_0}$ and $q \in L_2(\Omega)$. Similarly, we update the approximations as $\mathbf{u}_{k+1} = \mathbf{u}_k + \delta\mathbf{u}_k$ and $p_{k+1} = p_k + \delta p_k$ for $k = 0, 1, \cdots$ until convergence.

3. The Variable Viscosity Oseen-type Problem

Let $\mathbf{X}_{E_0}^h$ and P^h be finite dimensional subspaces of $\mathbf{H}_{E_0}^1$ and $L_2(\Omega)$, and let $\{\vec{\varphi}_i\}_{1\le i\le n_u}$ be the nodal basis of $\mathbf{X}_{E_0}^h$ and $\{\phi_i\}_{1\le i\le n_p}$ be the nodal basis of P^h. According to the Galerkin framework, the discrete corrections of the velocity and the pressure are represented as

$$\delta\mathbf{u}_h = \sum_{i=1}^{n_u} \delta\mathbf{u}_i\vec{\varphi}_i, \quad \delta\mathbf{p}_h = \sum_{i=1}^{n_p} \delta p_i\phi_i,$$

where n_u and n_p are the total number of degrees of freedom for the velocity and the pressure. The linear systems arising in Newton and Picard linearizations are of the form

$$\begin{bmatrix} F & B^T \\ B & O \end{bmatrix} \begin{bmatrix} \delta\mathbf{u}_h \\ \delta\mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \quad \text{or} \quad \mathcal{F}\mathbf{x} = \mathbf{b}, \tag{3.1}$$

where the system matrix $\mathcal{F} = \begin{bmatrix} F & B^T \\ B & O \end{bmatrix}$ is nonsymmetric and of a two-by-two block form. The matrix $B \in \mathbb{R}^{n_p\times n_u}$ corresponds to the (negative) divergence operator and B^T corresponds to the gradient operator (e.g., [14]). Here we assume either that the discrete LBB condition is satisfied, otherwise some stabilization is applied, resulting in a nonzero $(2,2)$ block. When comparing Newton and Picard linearization methods, the difference appears in the pivot block $F \in \mathbb{R}^{n_u\times n_u}$, which is of the form $F = A_\nu + \delta_1\widehat{A}_\nu + N + \delta_2\widehat{N}$. The Newton method corresponds to $\delta_1 = \delta_2 = 1$, while the Picard method corresponds to $\delta_1 = \delta_2 = 0$. Given the approximation $\mathbf{u}_h$, the entries of A_ν, $\widehat{A}_\nu$, N and $\widehat{N}$ are

$$A_\nu \in \mathbb{R}^{n_u\times n_u}, \quad [A_\nu]_{i,j} = \int_\Omega 2\nu(D_{\mathrm{II}}(\mathbf{u}_h))\mathbf{D}\vec{\varphi}_i : \mathbf{D}\vec{\varphi}_j, \tag{3.2a}$$

$$\widehat{A}_\nu \in \mathbb{R}^{n_u\times n_u}, \quad [\widehat{A}_\nu]_{i,j} = \int_\Omega 2\nu'(D_{\mathrm{II}}(\mathbf{u}_h))[\mathbf{D}\mathbf{u}_h : \mathbf{D}\vec{\varphi}_j][\mathbf{D}\mathbf{u}_h : \mathbf{D}\vec{\varphi}_i], \tag{3.2b}$$

$$N \in \mathbb{R}^{n_u\times n_u}, \quad [N]_{i,j} = \int_\Omega (\mathbf{u}_h\cdot\nabla\vec{\varphi}_j)\vec{\varphi}_i, \tag{3.2c}$$

$$\widehat{N} \in \mathbb{R}^{n_u\times n_u}, \quad [\widehat{N}]_{i,j} = \int_\Omega (\vec{\varphi}_j\cdot\nabla\mathbf{u}_h)\vec{\varphi}_i. \tag{3.2d}$$

In this paper the linear system (3.1) arising in Newton (2.3) or Picard method (2.5) is referred to as the Oseen-type problem with variable viscosity.

Computing the solutions of the linear systems in (3.1) is the kernel and most time-consuming part in the numerical simulations. Therefore, fast and reliable solution techniques are critical. As is well known, direct solution methods are highly robust with respect to both problem and discretization parameters, and are, therefore, a preferred choice in the numerical simulations performed by engineers and applied scientists. The limiting factors for the sparse direct solvers are most often the computer memory demands and the need to repeatedly factorize matrices, which are recomputed during the simulation process, as for instance, the Jacobians in nonlinear problems. For real industrial applications where the models are mostly in three space dimensions and result in very large scale linear systems of the type (3.1), rapidly convergent iterative methods, accelerated by a proper preconditioner become the methods of choice. In this work, we consider preconditioned Krylov subspace methods, see the books [1, 14, 32].

3.1. Preconditioning the variable viscosity Oseen-type problem

As already mentioned, the linear systems in (3.1) are of two-by-two block form, and how to precondition such systems have been intensively studied in the past decades. In this work we limit ourselves to preconditioners, based on approximate block factorizations of the coefficient matrix. The literature on this class of preconditioners is huge. We refer for more details to the articles [2–6,22,27], the surveys [7,10,11,34] and the books [1,14,32], with numerous references therein. In general, the exact factorization of a two-by-two block matrix reads

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & O \\ A_{21} & S \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ O & I_2 \end{bmatrix} = \begin{bmatrix} I_1 & O \\ A_{21}A_{11}^{-1} & I_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ O & S \end{bmatrix}, \tag{3.3}$$

where I_1 and I_2 are identity matrices of proper dimensions. The pivot block A_{11} is assumed to be nonsingular and $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the exact Schur complement matrix. In our case, $A_{11} = F$, $A_{12} = B^T$, $A_{21} = B$ and $A_{22} = O$. So, $S = -BF^{-1}B^T$.

As preconditioners for such matrices of two-by-two block form, block lower- or upper-triangular approximate factors are often used

$$\begin{bmatrix} \widetilde{A}_{11} & O \\ A_{21} & \widetilde{S} \end{bmatrix}, \qquad \begin{bmatrix} \widetilde{A}_{11} & A_{12} \\ O & \widetilde{S} \end{bmatrix}. \tag{3.4}$$

Here the matrix $\widetilde{A}_{11}$ denotes some approximation of A_{11}, given either in an explicit form or implicitly defined via an inner iterative solution method with a proper stopping tolerance. The matrix $\widetilde{S}$ is some approximation of the exact Schur complement S.

The results in [3] show that the quality of the preconditioners in (3.4) can be improved by making a sufficient number of inner iterations when implicitly approximating A_{11} and by choosing a sufficiently accurate approximation $\widetilde{S}$. The most challenging task turns out to be the construction of numerically and computationally efficient approximations of the Schur complement, which is in general dense and it is not practical to form it explicitly.

For the two-by-two block system arising in the incompressible Navier-Stokes equations with constant viscosity, several state-of-art approximations of the Schur complement are proposed and analysed, c.f., [8,9,13,15,17,20,24,28,30,31,35]. Among these preconditioning techniques, in this paper we choose the augmented Lagrangian (AL) method (see e.g., [2,8,9]). The reason is that the construction of the AL method and its variants is purely algebraic, and it can be straightforwardly used for the variable viscosity case. Besides, the AL method is fully independent of the mesh refinement and quite robust with respect to the viscosity number in the constant viscosity cases. For the incompressible non-Newtonian flows, the variation of the viscosity does effect the efficiency of the available preconditioners. In this work we choose the AL method, illustrate the impact of variations in viscosity on its performance and improve that further.

Following the AL framework, we first algebraically transform the system (3.1) into an equivalent one as follows

$$\begin{bmatrix} F + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \delta \mathbf{u}_h \\ \delta \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{f}} \\ \mathbf{g} \end{bmatrix} \quad \text{or} \quad \mathcal{F}_\gamma \mathbf{x} = \widehat{\mathbf{b}}, \tag{3.5}$$

where $\widehat{\mathbf{f}} = \mathbf{f} + \gamma B^T W^{-1}\ \mathbf{g}$, and $\gamma > 0$ and W are suitable scalar and matrix parameters. Clearly, the transformed system (3.5) has the same solution as (3.1) for any value of γ and any nonsingular matrix W.

The equivalent system (3.5) is what we intend to solve and the AL-type preconditioner proposed for $\mathcal{F}_\gamma$ in (3.5) is of a block lower-triangular form

$$\mathcal{P}_\gamma = \begin{bmatrix} F + \gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix}. \tag{3.6}$$

To distinguish from various modifications introduced later, the preconditioner $\mathcal{P}_\gamma$, where the original pivot blocks $F + \gamma B^T W^{-1} B$ and W are used, is referred to as the *ideal* AL preconditioner. It can be seen that the exact Schur complement $S_{\mathcal{F}_\gamma} = -B(F+\gamma B^T W^{-1} B)^{-1} B^T$ of the transformed matrix $\mathcal{F}_\gamma$ is approximated by $-\frac{1}{\gamma} W$. We analyse the ideal AL preconditioner using the technique in [9,17], for instance. Consider the following generalized eigenvalue problem

$$\mathcal{F}_\gamma \mathbf{v} = \lambda \mathcal{P}_\gamma \mathbf{v}. \tag{3.7}$$

We see that

$$\mathcal{P}_\gamma^{-1} \mathcal{F}_\gamma = \begin{bmatrix} I & (F + \gamma B^T W^{-1} B)^{-1} B^T \\ 0 & \gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T \end{bmatrix}.$$

Thus, the eigenvalues λ in (3.7) are either equal to 1 (with multiplicity equal to the dimension of $F \in \mathbb{R}^{n_u \times n_u}$ or coincide with those of the matrix $\gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T$. Applying Sherman-Morrison-Woodbury's formula to $(F + \gamma B^T W^{-1} B)^{-1}$, we obtain

$$\widetilde{Q} \equiv \gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T = \gamma Q - \gamma Q (I + \gamma Q)^{-1} \gamma Q,$$

where $Q = W^{-1} B F^{-1} B^T$. The matrix $B F^{-1} B^T$ is the negative Schur complement of the original system matrix $\mathcal{F}$ in (3.1). We state the following theorem, which has been shown in [18] and is included here only for completeness.

Theorem 3.1. *Let $\mu = a + i\,b$ be an eigenvalue of $Q = W^{-1} B F^{-1} B^T$, λ be an eigenvalue of the eigenproblem* (3.7) *and δ be an eigenvalue of the matrix $\widetilde{Q} = \gamma W^{-1} B (F + \gamma B^T W^{-1} B)^{-1} B^T$. Then the following holds:*

(1) The matrices Q and $\widetilde{Q}$ have the same eigenvectors and the eigenvalues of $\widetilde{Q}$ are equal to

$$\delta = \frac{\gamma \mu}{1 + \gamma \mu} = \frac{1}{1 + \dfrac{1}{\gamma \mu}}. \tag{3.8}$$

(2) The eigenvalues λ equal

$$\lambda = \begin{cases} 1, & \text{with multiplicity } n_u \\ \delta. \end{cases}$$

When $\gamma \to \infty$ all nonzero eigenvalues λ converge to 1.

(3) Assume that μ is bounded in a rectangular box, i.e., there exist constants $\{a_{\min}, a_{\max}, b_{\max}\}$, independent of the mesh size parameter h, such that

$$\left\{ \begin{array}{l} a_{\min} \le a \le a_{\max}, \\ \mid b \mid \le b_{\max}. \end{array} \right\}$$

Then λ is also bounded in a rectangular box with sizes, independent of h. Furthermore, there holds

$$\delta = 1 - \frac{1+\gamma a}{(1+\gamma a)^2+\gamma^2 b^2} + i\frac{\gamma b}{(1+\gamma a)^2+\gamma^2 b^2}. \tag{3.9}$$

For any $\gamma \geq 1$, and any value of a and b, we have

$$1 - \frac{1+\gamma|a|}{(1+\gamma a)^2+\gamma^2 b^2} < \mathcal{R}(\delta) < 1 \text{ and } |\mathcal{I}(\delta)| = \frac{\gamma|b|}{(1+\gamma a)^2+\gamma^2 b^2} < 1, \tag{3.10}$$

where $\mathcal{R}(\cdot)$ and $\mathcal{I}(\cdot)$ denote the real and the imaginary part of a complex number.

In this form, Theorem 3.1 is originally given in [18], where the viscosity is considered to be a function of space and time, as in multiphase flow problems. Although the viscosity is dependent of different parameters, Theorem 3.1 always holds true.

As mentioned, the transformation (3.5) is valid for any nonsingular matrix W. In practice W is often chosen to be the pressure mass matrix M as in [8], or to be the identity matrix as in [2,11]. Proposition 3.1 combined with Theorem 3.1 show that for the modified Bingham model with both Picard and Newton iterations, the ideal AL preconditioner with $W = M$ is independent of the mesh size. However, the independence on the variation of viscosity is not guaranteed. In order to achieve this objective, it is natural to let W incorporate some "information" of the variable viscosity. Therefore, we choose W also as the *scaled* pressure mass matrix, i.e.,

$$M_\nu = \{(M_\nu)_{i,j}\} \in \mathbb{R}^{n_p \times n_p}, \text{ with } (M_\nu)_{i,j} = (\nu^{-1}\phi_i, \phi_j). \tag{3.11}$$

The ideal AL preconditioner with $W = M_\nu$ is further analysed in Proposition 3.1. The conclusion is that the choice $W = M_\nu$ leads to the ideal AL preconditioner fully independent of the mesh size and rather robust with respect to (nearly independent of) the variation of the viscosity for both Picard and Newton iterations.

Proposition 3.1. *Consider the discrete modified Bingham model with $\nu_{min} = O(1)$ and $\nu_{max} = O(\varepsilon^{-1})$, $0 < \varepsilon \ll 1$, solved by either Picard or Newton iterations. Let μ denote an eigenvalue of $Q = W^{-1}BF^{-1}B^T$, M be the pressure mass matrix and M_ν be the scaled pressure mass matrix, defined as in (3.11). For the choices $W = M$ and $W = M_\nu$ the eigenvalues μ are contained in a rectangular box in the right half complex plane, with boundaries independent of the mesh size parameter h. More precisely, the following bounds for the real and imaginary parts of μ hold true:*

(i) For $W = M$,

$$\frac{c_0^2\nu_{\min}^2}{\nu_{\max}(\nu_{\min}^2+c_1^2)} \leq Re(\mu) \leq \frac{1}{\nu_{\min}}, \text{ and } |Im(\mu)| \leq \frac{1}{2\nu_{\min}}. \tag{3.12}$$

(ii) For $W = M_\nu$,

$$\frac{c_\nu^2\nu_{\min}^2}{\nu_{\min}^2+c_1^2} \leq Re(\mu) \leq C_\nu, \text{ and } |Im(\mu)| \leq \frac{C_\nu}{2}, \tag{3.13}$$

where in Picard iterations $C_\nu = d$ (d is the space dimension). The parameters c_0, c_1, c_ν and C_ν are positive constants that are independent of the discretization parameter h.

Proof. Let $S = BF^{-1}B^T$, $C = B(\frac{F^{-1}+F^{-T}}{2})B^T$ denote the symmetric part of S, and $R = B(\frac{F^{-1}-F^{-T}}{2})B^T$ denote the skew-symmetric part of S. Thus, $S = C + R$. By Bendixson's theorem, the eigenvalues μ satisfy

$$\min_{q_h \in P^h} \frac{(Cq_h, q_h)}{(Wq_h, q_h)} \leq \mathrm{Re}(\mu) \leq \max_{q_h \in P^h} \frac{(Cq_h, q_h)}{(Wq_h, q_h)}, \quad \text{and} \quad |\mathrm{Im}(\mu)| \leq \max_{q_h \in P^h} \frac{|(Rq_h, q_h)|}{(q_h, q_h)}.$$

We let $S' = B(A_\nu + \delta \widehat{A}_\nu)^{-1}B^T$, where A_ν and $\widehat{A}_\nu$ are defined in (3.2). In Picard iterations $\delta = 0$ while in Newton iterations $\delta = 1$.

For the symmetric part C, we have

$$\frac{(Cq_h, q_h)}{(Wq_h, q_h)} = \frac{(Cq_h, q_h)}{(S'q_h, q_h)} \frac{(S'q_h, q_h)}{(Wq_h, q_h)}, \tag{3.14}$$

and for the skew-symmetric part R

$$\frac{(Rq_h, q_h)}{(Wq_h, q_h)} = \frac{(Rq_h, q_h)}{(S'q_h, q_h)} \frac{(S'q_h, q_h)}{(Wq_h, q_h)}. \tag{3.15}$$

Theorem 3.2 in [18] proves that for $W = M$ in Picard iterations it holds

$$\frac{\nu_{\min}^2}{\nu_{\min}^2 + c_1^2} \leq \frac{(Cq_h, q_h)}{(S'q_h, q_h)} \leq 1 \text{ and } \left| \frac{(Rq_h, q_h)}{(S'q_h, q_h)} \right| \leq \frac{1}{2}, \tag{3.16}$$

where c_1 is a positive constant independent of the mesh size h. Following the proof of Theorem 3.2 in [18] it is easy to show that relation (3.16) holds for $W = M$ and $W = M_\nu$ in both Picard and Newton iterations. Then, it remains to bound the term $\frac{(S'q_h, q_h)}{(Wq_h, q_h)}$.

For Picard iterations with $W = M$, it has been proven in [16] that

$$\frac{c_0^2}{\nu_{\max}} \leq \frac{(S'q_h, q_h)}{(Mq_h, q_h)} \leq \frac{1}{\nu_{\min}}, \tag{3.17}$$

where c_0 is the constant in the LBB condition and is independent of h. Reference [16] considers only relation (3.17) for Picard's iterations. Proposition 4.1 in this paper shows that relation (3.17) also holds for Newton iterations. Then, by using relations (3.14)-(3.17) we have that for both Picard and Newton iterations with $W = M$

$$\frac{c_0^2 \nu_{\min}^2}{\nu_{\max}(\nu_{\min}^2 + c_1^2)} \leq \mathrm{Re}(\mu) \leq \frac{1}{\nu_{\min}}, \text{ and } |\mathrm{Im}(\mu)| \leq \frac{1}{2\nu_{\min}}.$$

For $W = M_\nu$ it is shown in [16] that for Picard iterations

$$c_\nu^2 \leq \frac{(S'q_h, q_h)}{(M_\nu q_h, q_h)} \leq d, \tag{3.18}$$

where d is the spacial dimension. The parameter c_ν is independent of the mesh size h, however, dependent of the regularization parameter ε, namely, the variation of the viscosity. Reference [16] shows the validity of relation (3.18) only for Picard iterations. In Section 4.1 of this paper we extend it to Newton iterations and get

$$c_\nu^2 \leq \frac{(S'q_h, q_h)}{(M_\nu q_h, q_h)} \leq C_\nu, \tag{3.19}$$

where c_ν is the same as in (3.18). The upper bound C_ν is a positive constant and independent of the mesh size h.

Similarly, with $W = M_\nu$ we have that for either Picard or Newton iterations

$$\frac{c_\nu^2 \nu_{\min}^2}{\nu_{\min}^2 + c_1^2} \leq \mathrm{Re}(\mu) \leq C_\nu, \text{ and } |\mathrm{Im}(\mu)| \leq \frac{C_\nu}{2},$$

where in Picard iterations C_ν is replaced by d. ■

Remark 3.1 For $W = M$, since $\nu_{\min} = O(1)$ and $\nu_{\max} = O(\varepsilon^{-1})$, we have that $\mathrm{Re}(\mu)_{\min} = O(\varepsilon)$, $\mathrm{Re}(\mu)_{\max} = O(1)$ and $|\mathrm{Im}(\mu)|_{\max} = O(1)$ in either Picard or Newton iterations. Thus, we see that the eigenvalues μ are independent of the mesh size h, however, dependent of ε, namely, variation of the viscosity.

Remark 3.2 For $W = M_\nu$, a deep discussion on c_ν in [16] shows that for Picard iterations only the smallest eigenvalue of $M_\nu^{-1} S'$ is of the order $O(\varepsilon)$, and the left eigenvalues are of the order $O(1)$. In Section 4.1 of this paper we consider the lower and upper bounds of (3.19) for Newton iterations, c_ν and C_ν, respectively. Section 4.1 of this paper shows that for Newton iterations c_ν is independent of the mesh size h and the regularization parameter ε. The upper bound C_ν is nearly independent of ε. Thus, based on (3.12)-(3.13) we can conclude that $W = M_\nu$ leads to a much more clustered eigenvalues μ of $Q = W^{-1} B F^{-1} B^T$ than $W = M$, for both Newton and Picard iterations.

Based on Theorem 3.1, Proposition 3.1 and Remarks 3.1-3.2, we conclude that the eigenvalues of the generalized eigenvalue problem (3.7), where the ideal AL preconditioner $\mathcal{P}_\gamma$ is constructed with $W = M_\nu$ are more clustered than those, corresponding to $\mathcal{P}_\gamma$ with $W = M$. In practice, with no hesitation, one can choose $W = \mathrm{diag}(M_\nu)$, since $c_m \mathrm{diag}(M_\nu) \leq M_\nu \leq C_m \mathrm{diag}(M_\nu)$ with positive mesh-independent constants c_m and C_m. The above analysis for $W = M_\nu$ holds also for $W = \mathrm{diag}(M_\nu)$. Thus, when using the AL preconditioner for the variable viscosity Oseen-type problem, we recommend to choose $W = \mathrm{diag}(M_\nu)$.

The second parameter in the AL scheme is the scalar γ. As Theorem 3.1 shows, for $\gamma \to \infty$ and for any nonsingular matrix W, all the eigenvalues of the preconditioned matrix $\mathcal{P}_\gamma^{-1} \mathcal{F}_\gamma$ cluster at one. This result means that for large values of γ and provided that we solve the sub-systems with the modified pivot block $F_\gamma = F + \gamma B^T W^{-1} B$ accurately enough, the ideal AL preconditioner ensures a very fast convergence, within a few iterations only. However, with increasing γ the modified pivot block F_γ becomes increasingly ill-conditioned and computing solutions of systems with F_γ becomes more and more difficult. Therefore, $\gamma = 1$ has been used in the numerical tests in many studies, for example [8, 17]. With $\gamma = 1$ the condition number of F_γ and the number of iterations by using ideal AL preconditioner are quite acceptable, see the numerical experiments in [8, 17]. In [18] the good properties of that choice are justified.

Although the matrices F and B are sparse, the modified pivot block F_γ is in general much denser. Furthermore, F_γ contains discretizations of mixed derivatives, and F_γ is not block-diagonal, as for Newtonian fluids. Besides, the mixed derivatives bring additional difficulties for the numerical solution methods. How to efficiently solve systems with F_γ in the AL framework is in general still an open question and more research efforts need to be invested here. In this work, we utilize the approach proposed in [9] and illustrate it for a problem in two space dimensions. In 2D, F is of the form $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$, where each block is square and of order

$n_u/2$. Denoting $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, we have

$$\begin{aligned} F_\gamma &= F + \gamma B^T W^{-1} B \\ &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} + \gamma \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} W^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\ &= \begin{bmatrix} F_{11} + \gamma B_1^T W^{-1} B_1 & F_{12} + \gamma B_1^T W^{-1} B_2 \\ F_{21} + \gamma B_2^T W^{-1} B_1 & F_{22} + \gamma B_2^T W^{-1} B_2 \end{bmatrix} \\ &:= \begin{bmatrix} F_{\gamma,11} & F_{\gamma,12} \\ F_{\gamma,21} & F_{\gamma,22} \end{bmatrix}. \end{aligned}$$

A possible approach, used in [9], is to approximate F_γ by a block lower-triangular matrix

$$\widetilde{F}_\gamma = \begin{bmatrix} \widetilde{F}_{\gamma,11} & O \\ F_{\gamma,21} & \widetilde{F}_{\gamma,22} \end{bmatrix},$$

and replacing F_γ by $\widetilde{F}_\gamma$ in the ideal AL preconditioner (3.6) we obtain the modified AL preconditioner as follows

$$\widetilde{\mathcal{P}}_\gamma = \begin{bmatrix} \widetilde{F}_\gamma & O \\ B & -\frac{1}{\gamma} W \end{bmatrix} = \begin{bmatrix} \widetilde{F}_{\gamma,11} & O & O \\ F_{\gamma,21} & \widetilde{F}_{\gamma,22} & O \\ B_1 & B_2 & -\frac{1}{\gamma} W \end{bmatrix}, \tag{3.20}$$

where the terms $\widetilde{F}_{\gamma,11}$ and $\widetilde{F}_{\gamma,22}$ denote approximations of $F_{\gamma,11}$ and $F_{\gamma,22}$, for instance, obtained via an inner iterative solution method with a proper stopping tolerance.

The modified AL preconditioner offers two main advantages compared to the ideal one. When solving systems with $\widetilde{F}_\gamma$ one needs to solve two sub-systems with $F_{\gamma,11}$ and $F_{\gamma,22}$. In this way, the size of the linear system to be solved is reduced. Besides, as already mentioned, there are approximations of mixed derivatives in F_γ, i.e., $F_{\gamma,21}$ and $F_{\gamma,12}$. This can be an obstacle when applying known solution techniques, such as algebraic multigrid (AMG) methods. Here we use AMG as a block solver and the details are presented in Section 5. A comparison between the performance of the multigrid solver, applied to the whole block F_γ and for the sub-blocks $F_{\gamma,11}$, $F_{\gamma,22}$ shows that the modified AL preconditioner is superior to the ideal AL preconditioner in terms of overall CPU time. Details are described in Section 5.

On the other hand, the performance of the modified AL preconditioner is dependent of the parameter γ. There exists an optimal value of γ which minimizing the number of iterations when using the modified AL preconditioner. For the case of constant viscosity, attempts to determine the optimal γ are found in [9]. Although some theory has been derived in [9], the optimal value turns out to be problem dependent and expensive to calculate. For non-Newtonian flows, we studied the effect of γ on the behaviour of the solver numerically. Results, not included here, show that a minimal number of iterations is obtained by choosing the value of γ to be 1. Therefore, for all numerical experiments in this paper, $\gamma = 1$ is used in the modified AL preconditioner.

So far, we only analyse the ideal AL preconditioner to check its robustness with respect to the mesh size h and the variation of the viscosity in the modified Bingham model. The theoretical analysis of the modified AL preconditioner, is a topic of separate research. In this paper the performance of the modified AL preconditioner for the variable viscosity Oseen-type problem is explored via thorough numerical experiments.

From the above discussion on M_ν and $\mathrm{diag}(M_\nu)$ in the modified Bingham model, we see that both M_ν and $\mathrm{diag}(M_\nu)$ are very good approximations of the matrix $BF^{-1}B^T$, which is the negative Schur complement of the original system matrix $\mathcal{F}$ in (3.1). In this way, it is not difficult to construct the preconditioner as given in (3.4)

$$\begin{bmatrix} \widetilde{F} & O \\ B & -\mathrm{diag}(M_\nu) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \widetilde{F} & B^T \\ O & -\mathrm{diag}(M_\nu) \end{bmatrix}$$

for the system matrix $\mathcal{F}$ in (3.1). The term $\widetilde{F}$ denotes the approximation defined via an inner iterative solution method. It seems that it is not necessary to use the AL transformation and the AL preconditioner. The comment on this issue is that the AL preconditioner is suitable for the Oseen-type problem when the convection term is relatively dominant. For the Oseen-type problem with constant viscosity, the AL preconditioner and its variation are very efficient for small values of the viscosity, see the numerical experiments in [8, 9, 17]. For large viscosity the diffusion part turns to be dominant and the problem becomes closer to the Stokes type. How to efficiently precondition the Stokes equations is simpler and has been introduced in earlier works, see, e.g. [14, 24]. For the considered modified Bingham model, the diffusion term is dominant compared to the convection term. For other convection-dominated non-Newtonian models, the ideal and modified AL preconditioner are very attractive, due to their purely algebraic construction and efficiency.

For diffusion-dominant non-Newtonian models, it is appropriate to consider the Stokes-type problem with variable viscosity. Up to the knowledge of the authors, efficient algorithms of the variable viscosity Stokes-type problems are only constructed for Picard iterations, c.f. [16, 30] In the next section we consider efficient preconditioners for both Picard and Newton iterations.

4. The Variable Viscosity Stokes-type Problem

As already stated in Section 2, at each nonlinear step the updates $(\delta\mathbf{u}_k, \delta p_k)$ are computed by solving the linear problem (2.3) via Newton method or the problem (2.5) via Picard method. At the $(k+1)$ nonlinear iteration, the velocity and the pressure are corrected as $\mathbf{u}_{k+1} = \mathbf{u}_k + \delta\mathbf{u}_k$ and $p_{k+1} = p_k + \delta p_k$. The above process continues until some convergence criterion is met. Since $\mathbf{u}_{k+1}$ and p_{k+1} are approximate solutions, when computing the updates $(\delta\mathbf{u}_k, \delta p_k)$, we could even drop the linearization terms coming from the convection terms in (2.3) and (2.5). Then, the linear problem in Newton method reads as follows:
Find $\delta\mathbf{u}_k \in \mathbf{H}^1_{E_0}$ and $\delta p_k \in L_2(\Omega)$ such that

$$\int_\Omega 2\nu(D_{\mathrm{II}}(\mathbf{u}_k))\mathbf{D}\delta\mathbf{u}_k : \mathbf{D}\mathbf{v}d\Omega + \int_\Omega 2\nu'(D_{\mathrm{II}}(\mathbf{u}_k))[\mathbf{D}\mathbf{u}_k : \mathbf{D}\delta\mathbf{u}_k][\mathbf{D}\mathbf{u}_k : \mathbf{D}\mathbf{v}]d\Omega$$
$$- \int_\Omega \delta p_k \ (\nabla \cdot \mathbf{v})d\Omega = R_k \tag{4.1a}$$
$$\int_\Omega q \ (\nabla \cdot \delta\mathbf{u}_k)d\Omega = P_k. \tag{4.1b}$$

The linear problem for the Picard method reads as follows: Find $\delta\mathbf{u}_k \in \mathbf{H}^1_{E_0}$ and $\delta p_k \in L_2(\Omega)$ such that

$$\int_\Omega 2\nu(D_{\rm II}(\mathbf{u}_k))\mathbf{D}\delta\mathbf{u}_k : \mathbf{D}\mathbf{v}d\Omega - \int_\Omega \delta p_k \ (\nabla\cdot\mathbf{v})d\Omega = R_k \tag{4.2a}$$

$$\int_\Omega q \ (\nabla\cdot\delta\mathbf{u}_k)d\Omega = P_k, \tag{4.2b}$$

for all $\mathbf{v}\in\mathbf{H}^1_{E_0}$ and $q\in L_2(\Omega)$. The residuals R_k, P_k are the same as given in (2.4), i.e.,

$$R_k = \int_\Omega \mathbf{f}\cdot\mathbf{v}d\Omega - \int_\Omega 2\nu(D_{\rm II}(\mathbf{u}_k))\mathbf{D}\mathbf{u}_k : \mathbf{D}\mathbf{v}d\Omega - \int_\Omega (\mathbf{u}_k\cdot\nabla\mathbf{u}_k)\cdot\mathbf{v}d\Omega + \int_\Omega p_k\nabla\cdot\mathbf{v}d\Omega$$

$$P_k = -\int_\Omega q \ (\nabla\cdot\mathbf{u}_k)d\Omega.$$

In this way, we see that the above iterative procedure involves the convection term in the right-hand side vector only. If the norm of the residuals R_k and P_k is smaller than the stopping tolerance, we can guarantee that the corresponding solutions satisfy the weak formulation (2.2).

After discretization with a stable FEM pair, the linear systems arising in (4.1) and (4.2) are of the form

$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix}\begin{bmatrix} \delta\mathbf{u}_h \\ \delta\mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \quad \text{or} \quad \mathcal{A}\mathbf{x} = \mathbf{b}, \tag{4.3}$$

where the pivot block A is symmetric positive definite (spd) and of the form $A = A_\nu + \delta_1\widehat{A}_\nu$. The terms A_ν and $\widehat{A}_\nu$ are the same as in (3.2). Newton's method corresponds to $\delta_1 = 1$ and Picard method – to $\delta_1 = 0$. The coefficient matrix $\mathcal{A}$ is symmetric but indefinite.

4.1. Preconditioning the variable viscosity Stokes-type problem

We refer to the problems (4.1) and (4.2) or their representations in matrix form (4.3) as the variable viscosity Stokes-type problems. Compared to the Oseen-type problem, the main benefit of solving the Stokes-type problem is that efficient approximations of the Schur complement in the Stokes-type problems are easier to construct, see e.g. [7, 10, 14] . We test two preconditioners for the Stokes-type problem–the block lower-triangular preconditioner and the SIMPLER preconditioner.

The block lower-triangular preconditioner is of the form $\mathcal{P}_L = \begin{bmatrix} \widetilde{A} & O \\ B & \widetilde{S} \end{bmatrix}$, where $\widetilde{A}$ denotes an approximation of the pivot block A defined via an inner iterative solution method, and the term $\widetilde{S}$ denotes an approximation of the exact Schur complement $S = -BA^{-1}B^T$. Still, the most difficult task is how to efficiently approximate the Schur complement. The Stokes problem arising in the incompressible Newtonian flows has been studied rather well and efficient approximations of S are well-known. For example, the pressure mass matrix M is a very efficient and numerically cheap approximation, see [24]. For the variable viscosity Stokes-type problem (4.3), it is natural to use the scaled pressure mass matrix M_ν or $\mathrm{diag}(M_\nu)$ as an approximation of the Schur complement, instead of the original pressure mass matrix M. The definition of M_ν is the same as given in Section 3. In earlier publications, related to the variable viscosity Stokes-type problem, the performance of $\widetilde{S} = -M$ and $\widetilde{S} = -M_\nu$ is only analysed for Picard iterations (c.f. [16]). In this paper the following theorem extends the analysis to the case of Newton iterations.

Proposition 4.1. *Consider the discrete modified Bingham model with $\nu_{min} = O(1)$ and $\nu_{max} = O(\varepsilon^{-1})$, $0 < \varepsilon \ll 1$, solved by Newton iterations. Let η denote an eigenvalue of $Q = \widetilde{S}^{-1}S$, M be the pressure mass matrix and M_ν be the scaled pressure mass matrix, defined as in (3.11). For the choices $\widetilde{S} = -M$ and $\widetilde{S} = -M_\nu$, the lower and upper bounds for the eigenvalues η are independent of the mesh size parameter h. More precisely, the following bounds for η hold true:*

(i) For $\widetilde{S} = -M$,

$$c_0^2 \nu_{\max}^{-1} \leq \eta \leq \nu_{\min}^{-1}. \tag{4.4}$$

(ii) For $\widetilde{S} = -M_\nu$,

$$c_\nu^2 \leq \eta \leq C_\nu, \tag{4.5}$$

where $C_\nu = \nu_{\min}^{-1}\nu_{\max}$. The parameters c_0, c_ν and C_ν are positive constants that are independent of the discretization parameter h.

Proof. For any $\mathbf{v}_h \in \mathbf{X}_{E_0}^h$ and $q_h \in P^h$, due to the definitions of matrices A, B and M, it holds

$$\begin{aligned}(BA^{-1}B^T q_h, q_h) &= (A^{-1}B^T q_h, B^T q_h) \\ &= \sup_{\mathbf{v}_h \in \mathbf{X}_{E_0}^h} \frac{(\mathbf{v}_h, B^T q_h)^2}{(A\mathbf{v}_h, \mathbf{v}_h)} = \sup_{\mathbf{v}_h \in \mathbf{X}_{E_0}^h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)^2}{(A\mathbf{v}_h, \mathbf{v}_h)}, \end{aligned} \tag{4.6a}$$

$$(Mq_h, q_h) = \|q_h\|^2, \tag{4.6b}$$

where

$$(A\mathbf{v}_h, \mathbf{v}_h) = \int_\Omega \nu \mathbf{D}\mathbf{v}_h : \mathbf{D}\mathbf{v}_h d\Omega + \int_\Omega \nu'(D_{\mathrm{II}}(\mathbf{u}))(\mathbf{D}\mathbf{u}_h : \mathbf{D}\mathbf{v}_h)^2 d\Omega. \tag{4.7}$$

In the modified Bingham model, namely, $\nu(D_{\mathrm{II}}(\mathbf{u})) = \nu_0 + \tau(D_{\mathrm{II}}(\mathbf{u}) + \varepsilon^2)^{-\frac{1}{2}} > 0$, the derivative $\nu'(D_{\mathrm{II}}(\mathbf{u}))$ in terms of $D_{\mathrm{II}}(\mathbf{u})$ is $\nu'(D_{\mathrm{II}}(\mathbf{u})) = -\frac{1}{2}\tau(D_{\mathrm{II}}(\mathbf{u}) + \varepsilon^2)^{-\frac{3}{2}} < 0$. Based on the formulas, we can get $\nu_{\min} = \nu_0$, $\nu_{\max} = O(\varepsilon^{-1})$, and $|\nu'|_{\min} = 0$, $|\nu'|_{\max} = O(\varepsilon^{-3})$. Compared to Picard iterations, the pivot matrix A is not necessarily positive definite any more in Newton iterations. For the considered modified Bingham model in this paper, we assume that A is positive definite.

Also, within the proof the following inequalities are needed, which have been proved in [16]

$$\|\nabla \cdot \mathbf{v}_h\|^2 \leq \|\mathbf{D}\mathbf{v}_h\|^2 \leq \|\nabla \mathbf{v}_h\|^2, \quad \forall \mathbf{v}_h \in \mathbf{X}_{E_0}^h.$$

Considering $\nu' < 0$, we directly get

$$\begin{aligned}(A\mathbf{v}_h, \mathbf{v}_h) &\leq \int_\Omega \nu \mathbf{D}\mathbf{v}_h : \mathbf{D}\mathbf{v}_h d\Omega \leq \nu_{\max} \int_\Omega \mathbf{D}\mathbf{v}_h : \mathbf{D}\mathbf{v}_h d\Omega \\ &= \nu_{\max}\|\mathbf{D}\mathbf{v}_h\|^2 \leq \nu_{\max}\|\nabla \mathbf{v}_h\|^2. \end{aligned} \tag{4.8}$$

Since the LBB condition is satisfied, i.e. there exists a mesh independent constant c_0 such that

$$c_0 \leq \inf_{q_h \in P^h} \sup_{\mathbf{v}_h \in \mathbf{X}_{E_0}^h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|q_h\|\|\nabla \mathbf{v}_h\|}.$$

Relation (4.6) together with estimate (4.8) and the LBB condition yield

$$O(\varepsilon)(Mq_h, q_h) = c_0^2 \nu_{\max}^{-1}(Mq_h, q_h) \leq (BA^{-1}B^T q_h, q_h).$$

Considering the minimal and maximal values of $|\nu'|$ and ν, we can get

$$(A\mathbf{v}_h, \mathbf{v}_h) \geq \nu_0 \|\mathbf{D}\mathbf{v}_h\|^2 = \nu_{\min} \|\mathbf{D}\mathbf{v}_h\|^2. \tag{4.9}$$

Using the Cauchy-Schwarz inequality we have

$$(q_h, \nabla \cdot \mathbf{v}_h)^2 \leq \|q_h\|^2 \|\nabla \cdot \mathbf{v}_h\|^2 \leq \|q_h\|^2 \|\mathbf{D}\mathbf{v}_h\|^2. \tag{4.10}$$

Estimates (4.9)-(4.10) together with relation (4.6) yield

$$(BA^{-1}B^T q_h, q_h) \leq \nu_{\min}^{-1} (Mq_h, q_h) = O(1)(Mq_h, q_h).$$

Thus, relation (4.4) is proved.

For $\widetilde{S} = -M_\nu$, it is easy to show that in Newton iterations

$$O(\varepsilon) M_\nu = c_0^2 \nu_{\min} \nu_{\max}^{-1} M_\nu \leq BA^{-1}B^T \leq \nu_{\min}^{-1} \nu_{\max} M_\nu = O(\varepsilon^{-1}) M_\nu. \tag{4.11}$$

This result can be obtained by using $\nu_{\min} M_\nu \leq M \leq \nu_{\max} M_\nu$. The inequality (4.11) does not give any improvement compared to the original pressure mass matrix M. As used in [16], here we also assume that there exist an coefficient c_ν such that

$$c_\nu \leq \inf_{q_h \in P^h} \sup_{\mathbf{v}_h \in \mathbf{X}_{E_0}^h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\nu^{-\frac{1}{2}} q_h\| \|\nu^{\frac{1}{2}} \mathbf{D}\mathbf{v}_h\|}.$$

using $(A\mathbf{v}_h, \mathbf{v}_h) \leq \|\nu^{\frac{1}{2}} \mathbf{D}\mathbf{v}_h\|^2$ and relation (4.6) we directly obtain

$$c_\nu^2 (M_\nu q_h, q_h) = c_\nu^2 \|\nu^{-\frac{1}{2}} q_h\|^2 \leq (BA^{-1}B^T q_h, q_h).$$

Then, in Newton iterations we can obtain

$$c_\nu^2 M_\nu \leq BA^{-1}B^T \leq C_\nu M_\nu,$$

where $C_\nu = \nu_{\min}^{-1} \nu_{\max}$. ∎

Remark 4.1. For $\widetilde{S} = -M$, Proposition 4.1 and the earlier reference [16] show that the inequalities (4.4) hold for both Picard and Newton iterations. Since $\nu_{\max} = O(\varepsilon^{-1})$ and $\nu_{\min} = O(1)$, the eigenvalues η are dependent of the regularization parameter ε, namely, the variation of the viscosity. Item (i) in Proposition 4.1 matches well with numerical experiments. We give the minimal and maximal eigenvalues of $M^{-1}BA^{-1}B^T$ in Table 5.5 of Section 5, where A is obtained at the last Newton iteration. As seen there, the maximal eigenvalue is a constant and the minimal one is of the order $O(\varepsilon)$.

Remark 4.2. From the results in [16] if is seen that for Picard iterations it holds

$$c_\nu^2 M_\nu \leq BA^{-1}B^T \leq dM_\nu,$$

where d is the spacial dimension and c_ν is the same as in (4.5). The coefficient c_ν is discussed in detail in [16] and the conclusions therein show that for Picard iterations the smallest eigenvalue of $M_\nu^{-1}BA^{-1}B^T$ is of the order $O(\varepsilon)$, and the left eigenvalues are of the order $O(1)$. For Newton iterations, the numerical results in Table 5.5 show that c_ν is constant, bounded independently of ε and h. For Newton iterations the upper bound in (4.5) is $C_\nu = \nu_{\min}^{-1} \nu_{\max} = O(\varepsilon^{-1})$. Results in Table 5.5 show that this bound is not optimal and C_ν is nearly independent of ε. The further analysis of c_ν and C_ν for Newton iterations is considered as a future research direction.

Combining the reference [16], Proposition 4.1 and Remarks 4.1–4.2, we conclude that the preconditioner $\mathcal{P}_L$ with $\widetilde{S} = -M_\nu$ is more efficient than $\widetilde{S} = -M$ for the Stokes-type problem (4.3) solved by either Newton or Picard iterations. Here, we again recommend to choose $\widetilde{S} = -\text{diag}(M_\nu)$ in practice with the same reason as given before.

In [16] at each preconditioning step the sub-systems with the pivot block A are solved by a direct method, which is clearly not suitable for large scale simulations in terms of CPU time and memory requirements. Here we suggest a computational improvement, based on the strategy of constructing the modified AL preconditioner, as described in (3.20), namely, in the two dimensional case, the pivot block $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is approximated by $\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & O \\ A_{21} & \widetilde{A}_{22} \end{bmatrix}$. The terms $\widetilde{A}_{11}$ and $\widetilde{A}_{22}$ denote approximations of A_{11} and A_{22} obtained by an inner iterative solution method with a proper stopping tolerance. In summary, the block lower-triangular preconditioner for the Stokes-type problem with variable viscosity is

$$\mathcal{P}_{Stokes} = \begin{bmatrix} \widetilde{A} & O \\ B & \widetilde{S} \end{bmatrix} = \begin{bmatrix} \widetilde{A}_{11} & O & O \\ A_{21} & \widetilde{A}_{22} & O \\ B_1 & B_2 & -\text{diag}(M_\nu) \end{bmatrix}. \tag{4.12}$$

SIMPLE (semi-implicit pressure linked equation) is used by Patanker [29] as an iterative method to solve the Navier-Stokes problem. The scheme belongs to the class of basic iterative methods and exhibits slow convergence. In [20, 35] SIMPLE and its variant SIMPLER are used as preconditioners in a Krylov subspace method to solve the incompressible Navier-Stokes equations, achieving in this way, a much faster convergence. SIMPLE and SIMPLER rely on an approximate block-factorization of saddle point matrices and due to their simplicity, remain attractive preconditioning techniques. We briefly describe both formulations for the Stokes matrix $\mathcal{A}$ in (4.3).

The SIMPLE preconditioner $\mathcal{P}_{SIMPLE}$ reads:

$$\mathcal{P}_{SIMPLE} = \begin{bmatrix} A & O \\ B & \widetilde{S} \end{bmatrix} \begin{bmatrix} I_1 & D^{-1}B^T \\ O & I_2 \end{bmatrix},$$

where D is the diagonal of the block A and $\widetilde{S} = -BD^{-1}B^T$. Solutions of systems with $\mathcal{P}_{SIMPLE}$ are straightforward, see Algorithm 4.1.

Algorithm 4.1 (Algorithm SIMPLE)
Given $\mathbf{y} = [\boldsymbol{y}_u; \boldsymbol{y}_p]$, $\mathbf{x} = \mathcal{P}_{SIMPLE}^{-1}\mathbf{y}$ *is found within the following steps.*

Step 1: Solve $A\boldsymbol{x}_u^\star = \boldsymbol{y}_u$

Step 2: Solve $\widetilde{S}\boldsymbol{x}_p = \boldsymbol{y}_p - B\boldsymbol{x}_u^\star$

Step 3: Compute $\boldsymbol{x}_u = \boldsymbol{x}_u^\star - D^{-1}B^T\boldsymbol{x}_p$

Step 4: Set $\boldsymbol{x} = [\boldsymbol{x}_u; \boldsymbol{x}_p]$

SIMPLER differs slightly from SIMPLE. It includes a pressure prediction step, see Algorithm 4.2.

Algorithm 4.2 (Algorithm SIMPLER)
Given $\mathbf{y} = [\boldsymbol{y}_u; \boldsymbol{y}_p]$, $\mathbf{x} = \mathcal{P}_{SIMPLER}^{-1}\mathbf{y}$ *is found within the following steps.*

Step 0: Solve $\widetilde{S}\boldsymbol{x}_p^\star = \boldsymbol{y}_p - BD^{-1}\boldsymbol{y}_u$

Step 1: Solve $A\boldsymbol{x}_u^\star = \boldsymbol{y}_u - B^T\boldsymbol{x}_p^\star$

Step 2: Solve $\widetilde{S}\delta\boldsymbol{x}_p = \boldsymbol{y}_p - B\boldsymbol{x}_u^\star$

Step 3: Update $\boldsymbol{x}_p = \boldsymbol{x}_p^\star + \delta\boldsymbol{x}_p$ *and* $\boldsymbol{x}_u = \boldsymbol{x}_u^\star - D^{-1}B^T\delta\boldsymbol{x}_p$

Step 4: Set $\boldsymbol{x} = [\boldsymbol{x}_u; \boldsymbol{x}_p]$

We see, that when applying $\mathcal{P}_{SIMPLE}$ and $\mathcal{P}_{SIMPLER}$, two solutions with $\widetilde{S}$ and one solution with A are required. Based on earlier experience, we modify $\mathcal{P}_{SIMPLE}$ and $\mathcal{P}_{SIMPLER}$ as follows:

(i) We choose $\widetilde{S} = -\mathrm{diag}(M_\nu)$. The choice is motivated by the previous analysis.

(ii) In Step 1, instead of solving systems with A, we approximate A as $\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & O \\ A_{21} & \widetilde{A}_{22} \end{bmatrix}$, where $\widetilde{A}_{11}$ and $\widetilde{A}_{22}$ indicate that we use an inner iterative solver with a proper stopping tolerance for the blocks A_{11} and A_{22}.

(iii) In order to improve the numerical stability, in this work the velocity block A is approximated by a diagonal matrix $D = \Sigma|A|$, where $\Sigma|A|$ denotes the row sum of absolute values of A.

As already discussed, the AL preconditioner is expected to work efficiently for the Oseen-type problem, typically when the convection term is dominant. Thus, in the variable viscosity Stokes-type problem we do not use the AL preconditioner.

5. Numerical Illustrations

We choose as a benchmark the well-known two-dimensional lid-driven cavity problem, equipped with the boundary conditions $u_1 = u_2 = 0$ for $x = 0, x = 1$ and $y = 0$; $u_1 = 1, u_2 = 0$ for $y = 1$. The problem is discretized using a uniform Cartesian mesh and the Q2-Q1 finite element pair. In this paper we consider the regularized Bingham model, i.e., $\nu(D_{\mathrm{II}}(\mathbf{u})) = \nu_0 + \tau(D_{\mathrm{II}}(\mathbf{u}) + \varepsilon^2)^{-\frac{1}{2}}$. We fix $\nu_0 = 1$ and vary the regularization parameter ε and the coefficient τ, as $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$, and $\tau = 1,\ 2.5$.

As already mentioned, we consider two nonlinear linearization methods - Picard and Newton iterations. In order to achieve fast convergence of the nonlinear solver, in this paper we utilize the combination of these two methods. On a given grid, for each pair of (ε, τ) Picard iterations are carried out first and terminated when the norm of the relative residual $\|[R_k; P_k]\|/\|[R_0; P_0]\|$ is decreased by two orders of magnitude, where (R_k, P_k) is defined in (2.4). Then the so-obtained solution is used as an initial guess for Newton iterations, repeated until the relative residual is redused by a factor of 10^{-6}. Due to the good initial guess, the Newton method takes only a few iterations to converge. Besides, the final solution with each pair of (ε, τ) on the current

Table 5.1: Oseen formulation, comparison between two choices of W in $\widetilde{\mathcal{P}}_\gamma$, $h = 1/32$.

	$\tau = 1$					$\tau = 2.5$				
ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$W = \text{diag}(M_\nu)$										
Picard iter.	7	16	32	43	58	13	31	59	72	98
Picard-GCR iter.	7	5	6	7	8	8	7	7	8	11
Newton iter.	4	4	6	12	9	4	5	6	8	9
Newton-GCR iter.	9	9	9	11	16	10	10	10	16	20
$W = \text{diag}(M)$										
Picard iter.	7	15	32	52	67	13	31	60	89	108
Picard-GCR iter.	10	20	41	229	>400	14	32	182	255	>400
Newton iter.	4	6	12	28	33	5	10	21	33	39
Newton-GCR iter.	20	52	90	197	>400	29	66	154	308	>400

grid is linearly interpolated to the next finer mesh, obtained by one regular mesh refinement. The interpolated solution is used as an initial guess for Picard iterations on the finer mesh. In this way, Picard iterations are independent of the mesh refinement.

Systems with the matrices arising in the linearized problems, i.e., $\mathcal{F}_\gamma$ in (3.5) and $\mathcal{A}$ in (4.3), are solved by a preconditioned iterative method, in this case the generalized conjugate residual method (GCR) [1] as it allows for variable preconditioning. The stopping tolerance for GCR is also relative and is denoted by $\epsilon_{\texttt{GCR}}$. In this paper we choose $\epsilon_{\texttt{GCR}} = 10^{-2}$. Decreasing the tolerance for GCR is not beneficial since the number of Picard and Newton iterations will not be significantly reduced.

The preconditioner for the Oseen-type problem is $\widetilde{\mathcal{P}}_\gamma$, defined in (3.6). With, $\gamma = 1$ and $W = \text{diag}(M_\nu)$ or $W = \text{diag}(M)$. The preconditioner for the Stokes-type problem is either $\mathcal{P}_{Stokes}$ in (4.12) or $\mathcal{P}_{SIMPLER}$ in Algorithm SIMPLER. When applying those preconditioners, we need to solve systems with the sub-blocks $F_{\gamma,11}$, $F_{\gamma,22}$ and A_{11} and A_{22}, respectively. In this paper this is done by an algebraic multigrid method, namely, agmg (see [23, 25, 26]). For agmg, the relative stopping tolerance is denoted by $\epsilon_{\texttt{agmg}}$. Unless stated otherwise, $\epsilon_{\texttt{agmg}} = 10^{-2}$. The implementation of agmg is in Fortran and a Matlab interface is provided. For nonsymmetric matrices agmg uses the GCR method accelerated by an algebraic multigrid preconditioner, based on aggregation techniques.

The presented GCR iterations are averaged over the total number of nonlinear iterations. The average GCR iterations per nonlinear iteration is denoted as Picard-GCR and Newton-GCR iterations. All results in this paper are carried out in Matlab 7.13 (R2011b), and performed on a Linux-64 platform with 4 Intel(R) Core i5 CPUs, 660@3.33GHz. The reported execution time is in seconds. Whenever agmg is used, the setup time is included in the reported time figures.

Tables 5.1-5.4 present results for the Oseen-type problem and Tables 5.5-5.8 for the Stokes-type problem.

Table 5.1 shows the performance of the modified AL preconditioner $\widetilde{\mathcal{P}}_\gamma$ with the two choices of W. From Table 5.1 we see that the choice of $W = \text{diag}(M_\nu)$ results in much less GCR iterations than the choice of $W = \text{diag}(M)$. Therefore, in the rest of the experiments for the ideal and modified AL preconditioners we fix $W = \text{diag}(M_\nu)$.

Table 5.2 shows a comparison between the ideal and the modified AL preconditioners, i.e., $\mathcal{P}_\gamma$ and $\widetilde{\mathcal{P}}_\gamma$. From Table 5.2 we see that as predicted the ideal AL preconditioner is independent

Table 5.2: Oseen formulation, comparison between $\widetilde{\mathcal{P}}_\gamma$ and $\mathcal{P}_\gamma$, $h = 1/64$.

	$\tau = 1$					$\tau = 2.5$				
ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$\widetilde{\mathcal{P}}_\gamma$										
Picard iter.	6	11	22	33	51	10	26	44	60	88
Picard-GCR iter.	6	7	9	10	11	7	7	9	10	12
Newton iter.	4	5	6	7	9	4	5	6	9	8
Newton-GCR iter.	9	9	9	15	22	10	10	11	13	19
Total solution time	3.8	6.9	11.8	24.6	53.8	5.5	12.5	24.9	48.5	120.9
$\mathcal{P}_\gamma$										
Picard iter.	6	11	22	33	48	10	26	44	60	82
Picard-GCR iter.	7	7	9	10	10	8	8	9	10	12
Newton iter.	3	4	6	6	7	4	5	6	5	5
Newton-GCR iter.	8	7	8	7	7	7	7	8	7	7
Total solution time	9.3	21.1	60,2	128.9	191.1	18.1	45.5	108.8	236.8	516.9

of the parameter ε in both Picard and Newton iterations. Using $\widetilde{\mathcal{P}}_\gamma$ the Picard-GCR iterations are independent of ε, but the number of Newton-GCR iterations increase a little for a smaller ε. The gain in total solution time by using the modified AL preconditioner $\widetilde{\mathcal{P}}_\gamma$ is substantial.

Table 5.3 illustrates the nonlinear and the average GCR iterations by using the modified AL preconditioner $\widetilde{\mathcal{P}}_\gamma$ in Oseen formulation. Also, the total computational time is reported. We see from Table 5.3 that as expected, the Picard-GCR and Newton-GCR iterations by using

Table 5.3: Oseen formulation with $\widetilde{\mathcal{P}}_\gamma$ as a preconditioner.

	$\tau = 1$					$\tau = 2.5$				
ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$h = 1/32$										
Picard iter.	7	16	32	43	58	13	31	59	72	98
Picard-GCR iter.	7	5	6	7	8	8	7	7	8	11
Newton iter.	4	4	6	12	9	4	5	6	8	9
Newton-GCR iter.	9	9	9	11	16	10	10	10	16	20
Total solution time	1.0	1.6	3.2	5.9	10.8	1.6	3.1	5.9	10.8	22.3
$h = 1/64$										
Picard iter.	6	11	22	33	51	10	26	44	60	88
Picard-GCR iter.	6	7	9	10	11	7	7	9	10	12
Newton iter.	4	5	6	7	9	4	5	6	9	8
Newton-GCR iter.	9	9	9	15	22	10	10	11	13	19
Total solution time	3.8	6.9	11.8	24.6	53.8	5.5	12.5	24.9	48.5	120.9
$h = 1/128$										
Picard iter.	5	9	17	27	40	9	21	39	59	84
Picard-GCR iter.	7	6	8	8	8	6	6	8	8	9
Newton iter.	4	5	7	6	11	4	5	6	6	9
Newton-GCR iter.	8	9	9	16	25	8	9	11	17	17
Total solution time	12.3	19.2	42.5	76.7	257.8	20.1	40.2	87.1	158.1	469.1

Table 5.4: Oseen formulation, agmg performance for $F_{\gamma,11}$, $\epsilon_{\texttt{agmg}} = 10^{-6}$ and random right-hand side vectors.

	Picard linearization		Newton's linearization	
	$\tau = 1$	$\tau = 2.5$	$\tau = 1$	$\tau = 2.5$
$h = 1/64$				
agmg setup time:	3.80e-2	4.00e-2	6.30e-2	5.60e-2
agmg solution time:	5.10e-2	4.90e-2	5.50e-2	6.70e-2
agmg total time:	8.90e-2	8.90e-2	1.18e-1	1.23e-1
agmg iter.:	17	16	19	19
Direct solver time:	1.35e-1	1.38e-1	1.14e-1	1.19e-1
$h = 1/128$				
agmg setup time:	1.85e-1	1.86e-1	1.90e-1	1.88e-1
agmg solution time:	2.13e-1	2.46e-1	2.45e-1	2.70e-1
agmg total time:	3.98e-1	4.32e-1	4.35e-1	4.58e-1
agmg iter.:	18	21	19	23
Direct solver time:	7.47e-1	7.47e-1	7.43e-1	7.47e-1
$h = 1/256$				
agmg setup time:	8.36e-1	8.36e-1	8.40e-1	8.36e-1
agmg solution time:	8.95e-1	1.03	1.03	1.13
agmg total time:	1.73	1.87	1.87	1.97
agmg iter.:	19	20	20	21
Direct solver time:	3.92	3.92	3.95	3.99

the modified AL preconditioner $\widetilde{\mathcal{P}}_\gamma$ are independent of the mesh refinement. Besides, Picard-GCR iterations are independent of the parameter ε. When decreasing the values of ε, a slight increase in the number of Newton-GCR iterations is observed, however, the number of Newton-GCR iterations is still very acceptable, even for the smallest ε tested. Due to the combination of Picard and Newton methods, we see that Newton linearization takes less than 10 iterations to converge for all pairs of (ε, τ). Thanks to interpolating the solution between meshes, the

Table 5.5: Stokes formulation: minimal and maximal eigenvalues of $M^{-1}BA^{-1}B^T$ and $M_\nu^{-1}BA^{-1}B^T$, $\tau = 1$ and A is obtained from the last Newton iteration.

ε	$\lambda_{\min}(M^{-1}BA^{-1}B^T)$	$\lambda_{\max}(M^{-1}BA^{-1}B^T)$	$\lambda_{\min}(M_\nu^{-1}BA^{-1}B^T)$	$\lambda_{\max}(M_\nu^{-1}BA^{-1}B^T)$
h=1/32				
10^{-2}	0.0017	0.4187	0.1246	0.9355
10^{-3}	1.7752e-4	0.4187	0.1204	0.9946
10^{-4}	1.7885e-5	0.4187	0.1200	1.3203
10^{-5}	1.8111e-6	0.4187	0.1200	1.7018
10^{-6}	1.8440e-7	0.4187	0.1200	1.7642
h=1/64				
10^{-2}	0.0017	0.4539	0.1427	0.9594
10^{-3}	1.7044e-4	0.4539	0.1401	1.0624
10^{-4}	1.7079e-5	0.4539	0.1396	1.1774
10^{-5}	1.7106e-6	0.4539	0.1396	1.5778
10^{-6}	1.7134e-7	0.4539	0.1396	2.0008

Table 5.6: Stokes formulation with $\mathcal{P}_{Stokes}$ as a preconditioner.

	$\tau = 1$					$\tau = 2.5$				
ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$h = 1/32$										
Picard iter.	7	16	32	43	58	13	31	59	72	97
Picard-GCR iter.	7	8	8	9	10	9	9	10	12	14
Newton iter.	3	4	6	9	7	4	4	5	9	11
Newton-GCR iter.	11	12	11	14	19	13	15	14	19	21
Total solution time	0.5	1.1	2.0	4.1	7.5	1.0	1.8	4.2	8.0	16.1
$h = 1/64$										
Picard iter.	6	11	22	34	51	10	26	44	61	88
Picard-GCR iter.	7	9	12	15	15	9	9	12	14	16
Newton iter.	3	4	6	6	8	4	4	6	9	7
Newton-GCR iter.	12	13	11	21	30	13	14	15	19	29
Total solution time	1.8	3.3	6.7	15.3	31.5	3.2	6.6	13.8	30.4	71.5
$h = 1/128$										
Picard iter.	5	9	17	27	40	9	21	39	59	78
Picard-GCR iter.	7	8	11	12	12	7	8	10	11	12
Newton iter.	3	4	7	6	9	3	5	4	6	9
Newton-GCR iter.	11	11	12	24	33	11	12	19	22	28
Total solution time	6.0	9.8	20.8	35.2	108.4	8.1	19.8	42.1	91.5	262.7

number of Picard iterations is independent of the mesh refinement. The total solution time illustrates that the computational procedure is nearly of the optimal complexity.

Table 5.7: Stokes formulation with $\mathcal{P}_{SIMPLER}$ as a preconditioner.

	$\tau = 1$					$\tau = 2.5$				
ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$h = 1/32$										
Picard iter.	7	16	32	43	59	13	31	59	72	101
Picard-GCR iter.	8	9	9	10	12	10	10	12	14	17
Newton iter.	3	4	6	10	8	4	4	5	9	11
Newton-GCR iter.	11	11	13	16	24	12	14	16	20	24
Total solution time	0.5	1.2	2.4	5.4	10.9	1.1	2.4	4.9	10.4	21.4
$h = 1/64$										
Picard iter.	6	11	22	34	53	10	26	44	61	92
Picard-GCR iter.	8	10	14	17	19	10	11	16	17	19
Newton iter.	3	4	6	6	9	4	4	5	8	6
Newton-GCR iter.	11	12	15	26	34	12	14	15	21	29
Total solution time	1.8	3.5	8.9	19.1	46.3	3.3	7.6	18.1	35.9	87.9
$h = 1/128$										
Picard iter.	5	9	17	27	40	9	21	39	59	78
Picard-GCR iter.	9	9	15	17	16	9	10	13	14	15
Newton iter.	3	5	7	7	10	3	5	4	6	10
Newton-GCR iter.	10	11	14	32	40	12	14	20	27	30
Total solution time	5.9	11.1	28.4	64.8	157.7	9.4	23.1	52.2	118.9	296.2

Table 5.8: Stokes formulation, agmg performance for A_{11}, $\epsilon_{\texttt{agmg}} = 10^{-6}$ and random right-hand side vectors.

	Picard linearization		Newton linearization	
	$\tau = 1$	$\tau = 2.5$	$\tau = 1$	$\tau = 2.5$
$h = 1/64$				
agmg setup time:	8.00e-3	8.00e-3	8.00e-3	9.00e-3
agmg solution time:	1.60e-2	1.80e-2	2.10e-2	2.20e-2
agmg total time:	2.40e-2	2.60e-2	2.90e-2	3.10e-2
agmg iter.:	16	18	21	20
Direct solver time:	3.32e-2	3.23e-2	3.44e-2	3.52e-2
$h = 1/128$				
agmg setup time:	2.40e-2	2.70e-2	2.60e-2	2.40e-2
agmg solution time:	8.20e-2	1.06e-1	9.00e-2	1.23e-1
agmg total time:	1.06e-1	1.33e-1	1.16e-1	1.47e-1
agmg iter.:	19	23	21	24
Direct solver time:	1.67e-1	1.57e-1	1.54e-1	1.57e-1
$h = 1/256$				
agmg setup time:	8.40e-2	9.45e-2	9.10e-2	8.50e-2
agmg solution time:	3.44e-1	4.45e-1	3.78e-1	5.29e-1
agmg total time:	4.28e-1	5.39e-1	4.69e-1	6.14e-1
agmg iter.:	19	22	21	22
Direct solver time:	8.35e-1	8.35e-1	8.35e-1	8.35e-1

Table 5.4 illustrates the performance of agmg itself. We set the parameter $\varepsilon = 10^{-4}$ to simulate a more difficult scenario. In the regularized Bingham model the extreme values of the viscosity are $\nu_{\min} = O(1)$ and $\nu_{\max} = O(\varepsilon^{-1})$. A smaller value of ε results in a larger variation in viscosity. Another reason we choose $\varepsilon = 10^{-4}$ is that this value is small enough to characterise the non-Newtonian flows. More details on this issue is presented at the end of this section. We see that the agmg solver is fully independent of the mesh size, the parameter τ and the different linearization methods. Also, we compare it with the 'backslash' direct sparse solver in Matlab. For the problem sizes we test, agmg already shows its superiority, that will be increasingly stronger for larger problem sizes and 3D problems.

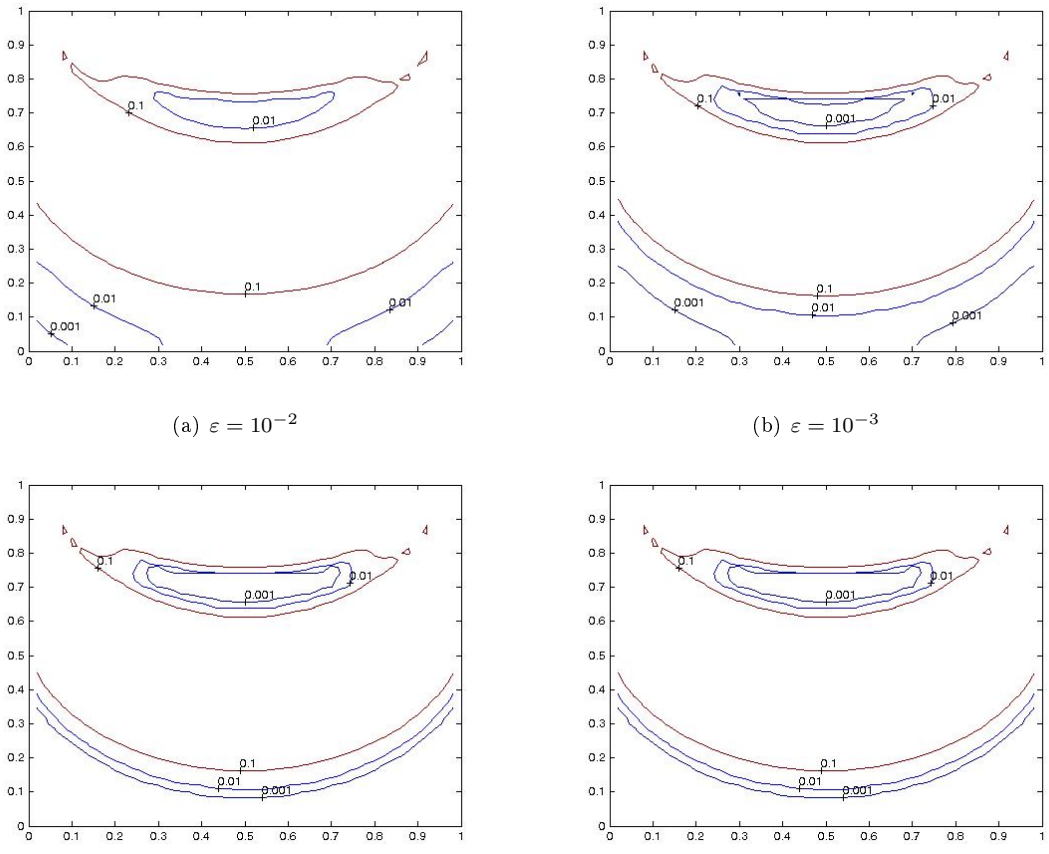
Next, in Tables 5.5-5.8 we illustrate the overall performance of the nonlinear and linear solvers for the variable viscosity Stokes-type problem. In Table 5.5 we calculate and present the smallest and largest eigenvalues of the matrices $M^{-1}BA^{-1}B^T$ and $M_\nu BA^{-1}B^T$ with varying ε. As seen, the smallest eigenvalue of $M^{-1}BA^{-1}B^T$ is of the order $O(\varepsilon)$ and the largest one is constant. This matches well with Proposition 4.1. For $M_\nu^{-1}BA^{-1}B^T$, the smallest eigenvalue is constant. The largest eigenvalue does increase a little with decreasing ε, but this increase is much less than that predicted as $O(\varepsilon^{-1})$ in Proposition 4.1. To explain this phenomenon, further analysis is needed. The extremal eigenvalues of $M^{-1}BA^{-1}B^T$ and $M_\nu BA^{-1}B^T$ are independent of the mesh size h, as seen in Table 5.5.

We see from Tables 5.6-5.7 that the linear Picard-GCR and Newton-GCR by using the preconditioners $\mathcal{P}_{Stokes}$ and $\mathcal{P}_{SIMPLER}$ are independent of the mesh refinement. The number of Picard-GCR iterations is independent of the parameter ε. For Newton-GCR iterations the independence of ε is loosen a little. Also, fast convergence rate of Newton iterations and independence of the mesh refinement for Picard iterations are achieved, due to the same reasons as

given in Oseen formulation. The total solution time by using $\mathcal{P}_{Stokes}$ preconditioner is slightly smaller than that with $\mathcal{P}_{SIMPLER}$. Still, with the two preconditioners $\mathcal{P}_{Stokes}$ and $\mathcal{P}_{SIMPLER}$, the computational procedure for the variable viscosity Stokes-type problem is of optimal complexity.

The efficiency of `agmg` for A_{11} is presented in Table 5.8. In the Stokes-type problem, since the sub-blocks A_{11} and A_{22} are `spd`, `agmg` uses the conjugate gradient (`CG`) Krylov subspace method accelerated by the multigrid preconditioner. Here, the `agmg` solver is also fully independent of the mesh size, the parameter τ and different linearization methods. The superiority to the direct method is exhibited too.

The comparison between the behaviour of the numerical solution methods for the Oseen- and Stokes-type problems (Tables 5.3, 5.6 and 5.7) shows that in both cases the number of nonlinear Picard and Newton iterations is the same. The explanation for this effect is that the diffusion term is dominant for the considered Bingham model. For other convection-dominant non-Newtonian models, the number of nonlinear iterations can be reduced by solving the Oseen-type problem. In those applications the introduced solution algorithms for the Oseen-type problem are applicable and attractive.



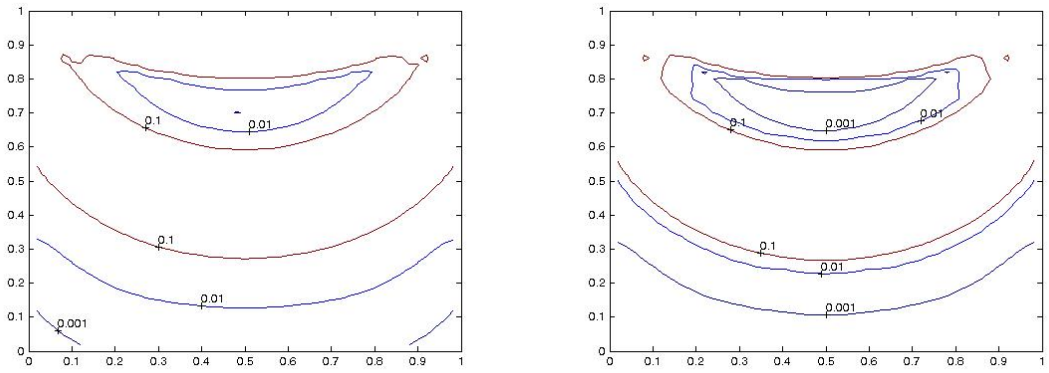
(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-3}$

(c) $\varepsilon = 10^{-4}$ (d) $\varepsilon = 10^{-5}$

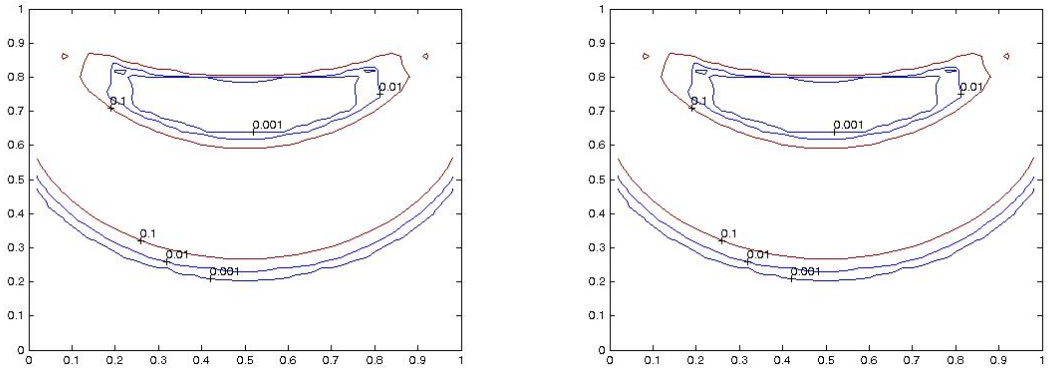
Fig. 5.1. Computed isolines for $(D_{\mathrm{II}}(\mathbf{u}))^{\frac{1}{2}} = \{10^{-1},\ 10^{-2},\ 10^{-3}\}$ with $\tau = 1$.

For the two problems, the overall computational time for the Stokes-type problem is only half of that for the Oseen-type problem. Further, the sub-blocks A_{11} and A_{22} are `spd` and `agmg` uses the `CG` method, which is numerically cheaper than the `GCR` method used for $F_{\gamma,11}$ and $F_{\gamma,22}$. Also, the blocks A_{11} and A_{22} are sparser than $F_{\gamma,11}$ and $F_{\gamma,22}$, and the sparsity is another reason making `agmg` to work more efficiently for the Stokes formulation. Comparing the results in Table 5.4 and 5.8, we see that the overall computational time of `agmg` for the Stokes-type problem is reduced about three times, compared to that for the Oseen-type problem.

Finally, we include some plots of the numerically computed solution. Determining the rigid regions of the viscoplastic flow, formally regions where $D_{\mathrm{II}}(\mathbf{u}) = 0$, is the most challenging task from modeling point of view. However, when a regularized model is used the condition $D_{\mathrm{II}}(\mathbf{u}) = 0$ does not hold exactly. In practice one needs to choose the regularization parameter ε as small as possible. On the other hand, small values of the regularization parameter ε lead to more computational work, see the nonlinear iterations and the total solution time in the previous tables. To give an insight regarding reasonable values of ε which can well predict the rigid regions, Figures 5.1-5.2 show the computed isolines of $(D_{\mathrm{II}}(\mathbf{u}))^{\frac{1}{2}}$ for $\varepsilon \in \{10^{-2},\ 10^{-3},\ 10^{-4},\ 10^{-5}\}$. These figures appear to be nearly identical with those in [12,16] and we see that for $\varepsilon \approx 10^{-4}$ the computed results give a fairly good prediction of the rigid regions.



(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-3}$



(c) $\varepsilon = 10^{-4}$ (d) $\varepsilon = 10^{-5}$

Fig. 5.2. Computed isolines for $(D_{\mathrm{II}}(\mathbf{u}))^{\frac{1}{2}} = \{10^{-1},\ 10^{-2},\ 10^{-3}\}$ with $\tau = 2.5$.

The relatively large values, i.e., $\varepsilon \geq 10^{-3}$ are not small enough to recover the viscoplastic properties.

6. Conclusions and Future Work

In this paper we consider fast and reliable numerical solution methods for the incompressible non-Newtonian Navier-Stokes equations. Among the several non-Newtonian fluid models here we limit ourselves to the regularized Bingham model. When linearizing the governing nonlinear equations, Oseen- or Stokes-type problems arise. In both cases, the coefficient matrices are of a two-by-two block form. Numerically and computationally efficient preconditioners for the so-arising systems are the main concerns in this paper. Various preconditioners are analysed, namely, the modified augmented Lagrangian preconditioner for the Oseen-type problem and the block lower-triangular and the SIMPLER preconditioners for the Stokes-type. Numerical experiments show that all the preconditioners are independent of the mesh size, and are rather robust with respect to the parameters in the Bingham model. Due to their algebraic constructions, the tested preconditioning techniques are straightforwardly applicable for other non-Newtonian fluid models. A detailed study on their performance in other applications is subject to a future research.

How to accelerate the convergence of the nonlinear solver is also studied in this paper. Numerical experiments show that a combination of Picard and Newton methods and the interpolation technique used in this paper are successful.

References

[1] O. Axelsson, Iterative Solution Methods, Cambridge University Press, Cambridge, 1994.

[2] O. Axelsson and R. Blaheta, Preconditioning of matrices partitioned in two-by-two block form: Eigenvalue estimates and Schwarz DD for mixed FEM, *Num. Lin. Alg. Appl.*, **17** (2010), 787-810.

[3] O. Axelsson and M. Neytcheva, A General Approach to Analyse Preconditioners for Two-by-Two Block Matrices, *Num. Lin. Alg. Appl.*, 2011, doi: 10.1002/nla.830.

[4] O. Axelsson and M. Neytcheva, Eigenvalue estimates for preconditioned saddle point matrices, *Num. Lin. Alg. Appl.*, **13** (2006), 339-360.

[5] Z.-Z. Bai, Structured preconditioners for nonsingular matrices of block two-by-two structures, *Math. Comput.*, **75** (2006), 791-815.

[6] Z.-Z. Bai and M.K. Ng, On inexact preconditioners for nonsymmetric matrices, *SIAM J. Sci. Comput.*, **26** (2005), 1710-1724.

[7] M. Benzi, Preconditioning techniques for large linear systems, *J. Comput. Phys.*, **182** (2002), 418-477.

[8] M. Benzi and M.A. Olshanskii, An augmented Lagrangian-based approach to the Oseen problem, *SIAM J. Sci. Comput.*, **28** (2006), 2095-2113.

[9] M. Benzi, M.A. Olshanskii and Z. Wang, Modified augmented Lagrangian preconditioners for the incompressible Navier-Stokes equations, *Int. J. Numer. Meth. Fluids.*, **66** (2011), 486-508.

[10] M. Benzi, G.H. Golub and J. Liesen, Numerical solution of saddle point problems. *Acta Numer.*, **14** (2005), 1-137.

[11] A.C. de Niet and F.W. Wubs, Two preconditioners for saddle point problems in fluid flows, *Int. J. Numer. Meth. Eng.*, **54** (2007), 355-377.

[12] E.J. Dean, R. Glowinski and G. Guidoboni, On the numerical simulation of Bingham viscoplastic flow: old and new results, *J. Non-Newtonian Fluid Mech.*, **142** (2007), 36-62.

[13] H.C. Elman and D.J. Silvester, Fast nonsymmetric iterations and preconditioning for Navier-Stikes equations, *SIAM J. Sci. Comput.*, **17** (1996), 33-46.

[14] H.C. Elman, D.J. Silvester and A.J. Wathen, Finite Element and Fast Iterative Solvers: with Application in Incompressible Fluid Dynamics, Oxford University Press, Oxford, UK, 2005.

[15] H.C. Elman, D.J. Silvester and A.J. Wathen, Performance and analysis of saddle point preconditioners for the discrete steady-state Navier-Stokes equations, *Numer. Math.*, **90** (2002), 665-688.

[16] P.P. Grinevich and M.A. Olshanskii, An iterative method for the Stokes-type problem with variable viscosity, *SIAM J. Sci. Comput.*, **31** (2009), 3959-3978.

[17] X. He, M. Neytcheva and S.S. Capizzano, On an Augmented Lagrangian-Based Preconditioning of Oseen Type Problems, *BIT Numer. Math.*, **51** (2011), 865-888.

[18] X.He and M. Neytcheva, Preconditioning the incompressible Navier-Stokes equations with variable viscosity, *J. Comput. Math.*, **30** (2012), 461-482.

[19] J. Hron, A. Quazzi and S. Turek, A computational comparison of two FEM solvers for nonlinear incompressible flow, *Lect. Notes Comp. Sci.*, **35** (2003), 87–109.

[20] C. Li and C. Vuik, Eigenvalue analysis of the SIMPLE preconditioning for incompressible flow, *Numer. Linear Algebra Appl.*, **11** (2004), 511-523.

[21] J. Malék, J. Nečas and K. R. Rajagopal, Global existence of solutions for flows of fluids with pressure and shear dependent viscosities, *Appl. Math. Lett.*, **15** (2002), 961-967.

[22] M.F. Murphy, G.H. Golub and A. J. Wathen, A note on preconditioning for indefinite linear systems, *SIAM J. Sci. Comput.*, **21** (2000), 1969-1072.

[23] A. Napov and Y. Notay, An algebraic multigrid method with guaranteed convergence rate, *SIAM J. Sci. Comput.*, **34** (2012), A1079-A1109.

[24] M. Neytcheva, M. Do-Quang and X. He, Element-by-element Schur complement approximations for general nonsymmetric matrices of two-by-two block form, *Lect. Notes Comput. Sc.*, **5910** (2009), 108-115.

[25] Y. Notay, An aggregation-based algebraic multigrid method, *Electron. Trans. on Numer. Anal.*, **37** (2010), 123-146.

[26] Y. Notay, Aggregation-based algebraic multigrid for convection-diffusion equations, *SIAM J. Sci. Comput.*, **34** (2012), A2288-A2316.

[27] Y. Notay, A new analysis of block preconditioners for saddle point problems, Report GANMN 13-01, University Libre de Bruxelles, Brussels, Belgium, 2013.

[28] M.A. Olshanskii and Y. Vassilevski, Pressure Schur complement preconditioners for the discrete Oseen problem, *SIAM J. Sci. Comput.*, **29** (2007), 2686-2704.

[29] S.V. Patankar, Numerical heat transfer and fluid flow, McGraw-Hill, New York, 1980.

[30] M. ur Rehman, T. Geenen, C. Vuik, G. Segal and S.P. MacLachlan, On iterative methods for the incompressible Stokes problem, *Int. J. Num. Meth. in Fluids*, **65** (2011), 1180-1200.

[31] T. Rusten and R. Winther, A preconditioned iterative method for saddle point problems, *SIAM J. Matrix Anal. Appl.*, **13** (1992), 887-904.

[32] Y. Saad, Iterative Methods for Sparse Linear Systems (2nd edn), SIAM: Philadelphia, PA, 2003.

[33] D. Schaeffer, Instability in the evolution equations describing incompressible granular flow, *J. Diff. Eq.*, **66** (1987), 19-50.

[34] A.Segal, M. ur Rehman and C. Vuik, Preconditioners for the incompressible Navier-Stokes equations, *Numer. Math. Theor. Meth. Appl.*, **3** (2010), 245-275.

[35] C. Vuik, A. Saghir and G.P. Boerstoel, The Krylov accelerated SIMPLE(R) method for flow problems in industrial furnaces, *Int. J. Numer. Meth. Fluids*, **33** (2000), 1027-1040.