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CONVERGENCE OF FINITE VOLUME SCHEMES FOR HAMILTON-JACOBI EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS*

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Abstract

We study numerical methods for time-dependent Hamilton-Jacobi equations with weak Dirichlet boundary conditions. We first propose a new class of abstract monotone approximation schemes and get a convergence rate of 1/2 . Then, according to the abstract convergence results, by newly constructing monotone finite volume approximations on interior and boundary points, we obtain convergent finite volume schemes for time-dependent Hamilton-Jacobi equations with weak Dirichlet boundary conditions. Finally give some numerical results.

Mathematics subject classification: 65N30.
Key words: Hamilton-Jacobi equations, Dirichlet boundary conditions, Finite volume, Monotone schemes.

1. Introduction

Hamilton-Jacobi equations arise in many areas of applied mathematics, like optimal control, differential games, geometric optics, seismic wave propagation, obstacle navigation, path planning and financial mathematics, among many others. They also appear when modeling evolving interfaces in geometry, fluid mechanics, computer vision and materials science. In general, these nonlinear first order PDEs cannot be solved analytically. The solutions usually develop singularities in their derivatives even with smooth initial conditions. In these cases, the solutions do not satisfy the equation in the classical sense. The weak solution that is usually sought is called the viscosity solution [12]. Numerically, in general, one looks for a consistent and monotone scheme to construct approximate viscosity solutions. In this paper, we study the following Dirichlet type boundary value problem of Hamilton-Jacobi equation.

$$\begin{cases} u_t + H(x, Du) = 0, & \Omega \times (0, T] \\ u(x,0) = u_0(x), & x \in \overline{\Omega} \\ u(x,t) = g(x,t), & x \in \partial\Omega,\ 0 < t \leq T, \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^N$ is an open and bounded set and $g(x,0) = u_0(x), x \in \partial\Omega$.

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The viscosity solution of Hamilton-Jacobi equation with weak Dirichlet boundary condition (1.1) is defined as follows.

Definition 1.1. *A locally bounded function u is a viscosity subsolution of* (1.1) *if and only if, for any $\varphi \in C^1\big(\overline{\Omega} \times [0,T]\big)$, if $u^* - \varphi$ has a local maximum at $(x_0,t_0) \in \overline{\Omega} \times (0,T]$, then*

$$\begin{cases} \varphi_t(x_0,t_0) + H\big(x_0, D\varphi(x_0,t_0)\big) \le 0, \ x_0 \in \Omega \\ \min\Big\{u^*(x_0,t_0) - g(x_0,t_0), \varphi_t(x_0,t_0) + H\big(x_0, D\varphi(x_0,t_0)\big)\Big\} \le 0, \ x_0 \in \partial\Omega \\ u^*(x,0) \le u_0(x), \ \forall x \in \overline{\Omega}. \end{cases} \tag{1.2}$$

A locally bounded function v is a viscosity supersolution of (1.1) *if and only if, for any $\varphi \in C^1\big(\overline{\Omega} \times [0,T]\big)$, if $v_* - \varphi$ has a local minimum at $(x_0,t_0) \in \overline{\Omega} \times (0,T]$, then*

$$\begin{cases} \varphi_t(x_0,t_0) + H\big(x_0, D\varphi(x_0,t_0)\big) \ge 0, \ x_0 \in \Omega \\ \max\Big\{v_*(x_0,t_0) - g(x_0,t_0), \varphi_t(x_0,t_0) + H\big(x_0, D\varphi(x_0,t_0)\big)\Big\} \ge 0, \ x_0 \in \partial\Omega \\ v_*(x,0) \ge u_0(x), \ \forall x \in \overline{\Omega}. \end{cases} \tag{1.3}$$

A function u is a viscosity solution of (1.1) *if it is both a sub- and a supersolution. The notations $\cdot^*$ and $\cdot_*$ refer to the upper and lower semicontinuous envelope, that is,*

$$u^*(x,t) = \limsup_{y \in \overline{\Omega} \to x, s \in [0,T] \to t} u(y,s) \text{and} v_*(x,t) = \liminf_{y \in \overline{\Omega} \to x, s \in [0,T] \to t} v(y,s).$$

Existence and uniqueness(comparison principle) of regular(uniformly continuous or Lipschitz continuous) viscosity solutions of time-dependent Hamilton-Jacobi equations are obtained essentially under two types of assumptions on Hamiltonian H, that is, Lipschitz continuity of H or uniform coercivity of mapping $p \to H(x,p)$ (see, e.g. [4,5,9,12,22]).

The problem of constructing convergent monotone approximation schemes for viscosity solutions of Hamilton-Jacobi and fully nonlinear second order partial differential equations has been considered by several authors (see e.g. [2,6,13,17-19,21]). In [11] obtained the first local a posteriori error estimate for time-dependent Hamilton-Jacobi equations, which becomes a tool for devising adaptive algorithms with error control. Especially in [22], a class of monotone numerical schemes for Hamilton-Jacobi equations with Dirichlet boundary conditions was studied. Up to our knowledge, it is the first time where proposed convergent approximation schemes for time-dependent Hamilton-Jacobi equations with Dirichlet boundary conditions and moreover, the problem of constructing numerically useful schemes has not been deeply studied. On the other hand, as pointed out in [22], the problem of constructing numerically useful schemes by using the abstract scheme proposed in the paper turned out to have some theoretical difficulties because how to interpolate between the boundary and interior grid points is generally not obvious and therefore, finding a scheme with the right properties in the vicinity of the boundary is a theoretical, but not computational problem. In Sect. 2 we provide a new class of abstract monotone schemes for equation (1.1) and get a convergence rate of $1/2$ under some usual assumptions on the data and assumption on monotonicity of H at the boundary $\partial\Omega$, that was proposed in [22]. Based on the obtained abstract results, we can construct numerically useful convergent schemes for time-dependent Hamilton-Jacobi equations with Dirichlet boundary conditions. In Sect. 3 we propose and analyze numerical schemes on equation (1.1) by finite volume method.