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CONVERGENCE ANALYSIS FOR THE ITERATED DEFECT CORRECTION SCHEME OF FINITE ELEMENT METHODS ON RECTANGLE GRIDS*

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Abstract

This paper develops a new method to analyze convergence of the iterated defect correction scheme of finite element methods on rectangular grids in both two and three dimensions. The main idea is to formulate energy inner products and energy (semi)norms into matrix forms. Then, two constants of two key inequalities involved are min and max eigenvalues of two associated generalized eigenvalue problems, respectively. Local versions on the element level of these two generalized eigenvalue problems are exactly solved to obtain sharp (lower) upper bounds of these two constants. This and some essential observations for iterated solutions establish convergence in 2D and the monotone decreasing property in 3D. For two dimensions the results herein improve those in literature; for three dimensions the results herein are new. Numerical results are presented to examine theoretical results.

Mathematics subject classification: 65N30, 65N15, 35J25.
Key words: Petrov-Galerkin method; iterated defect correction scheme; convergence, eigenvalue problem.

1. Introduction

Let Ω be a polygonal domain in $\mathbb{R}^d, d = 2, 3$ with boundary $\Gamma := \partial\Omega$. We consider the iterated defection correction scheme of finite element methods proposed in [7] for the following second order elliptic equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $f \in L^2(\Omega)$.

For a measurable set $G \subset \Omega$, let $(\cdot, \cdot)_{L^2(G)}$ and $||\cdot||_{L^2(G)}$ denote the inner product and the norm in $L^2(G)$, and if $G = \Omega$, we drop the index $L^2(\Omega)$ for simplicity. Then the weak formulation of the problem (1.1) reads: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \text{ for any } v \in H_0^1(\Omega) \tag{1.2}$$

with $a(u, v) := \int_\Omega \nabla u \cdot \nabla v dx$, where the symbol $\cdot$ is the inner product in the Euclidean space $\mathbb{R}^d, d = 2, 3$.

To consider the iterated defection correction scheme of finite element methods proposed in [7], let $\mathcal{M}_h$ be a shape regular triangulation of Ω into rectangles and $\mathcal{T}_h$ be the Green refinement of $\mathcal{M}_h$: for two dimensions, each element in $\mathcal{M}_h$ is refined into four congruence rectangles; for three dimensions, each element in $\mathcal{M}_h$ is refined into eight congruence rectangles. Throughout

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this paper, $h := \max_{M\in\mathcal{M}_h} h_M$ where h_M denotes the diameter of element M. Given $\omega \subset \Omega$ and integer $k \geq 0$, let $Q_k(\omega)$ denote the space of polynomials of degree $\leq k$ in each variable over ω. The conforming bilinear/trilinear finite element space over $\mathcal{T}_h$ and biquadratic/tri-quadratic finite element space over $\mathcal{M}_h$ are defined as, respectively,

$$V_{1,h} := \Big\{v \in H_0^1(\Omega), v|_K \in Q_1(T) \text{ for any } T \in \mathcal{T}_h\Big\},$$
$$V_{2,h} := \Big\{v \in H_0^1(\Omega), v|_K \in Q_2(M) \text{ for any } M \in \mathcal{M}_h\Big\}.$$

Given any $v \in H_0^1(\Omega) \cap C(\bar{\Omega})$, define the interpolation $\Pi_i v \in V_{i,h}, i = 1, 2$, by

$$(\Pi_i v)(\mathcal{P}) = v(\mathcal{P}) \text{ for any vertex } \mathcal{P} \text{ of } \mathcal{T}_h.$$

The Petrov-Galerkin method reads: Find $U_h \in V_{2,h}$ such that

$$a(U_h, v_h) = (f, v_h) \text{ for any } v_h \in V_{1,h}. \tag{1.3}$$

Given some approximation $u_{i,h} \in V_{1,h}$ to the solution u of (1.1), the iterated defect correction scheme is: Find $u_{i+1,h} \in V_{1,h}$ such that

$$\begin{aligned} a(u_{i+1,h}, v_h) = a(u_{i,h}, v_h) - (a(\Pi_2 u_{i,h}, v_h) - (f, v_h)) \\ \text{for any } v_h \in V_{1,h}, \quad i = 0, 1, \cdots, \end{aligned} \tag{1.4}$$

Usually, the initial approximation $u_{0,h}$ of algorithm (1.4) can be taken as the solution of the following discrete problem: Find $u_{0,h} \in V_{1,h}$ such that

$$a(u_{0,h}, v_h) = (f, v_h) \text{ for any } v_h \in V_{1,h}. \tag{1.5}$$

Concerning algorithm (1.4), one problem is whether problem (1.3) is well-posed while another problem is whether iterated solutions $u_{i,h}$ converges to the solution U_h of problem (1.3) in the following sense:

$$\lim_{i\to\infty} \|\nabla(U_h - \Pi_2 u_{i,h})\| = 0. \tag{1.6}$$

For uniform meshes in two dimensions, such convergence was analyzed in [3, 14, 15, 17] based on superconvergence of finite element solutions. For more general triangular meshes, the well-posedness of problem (1.3) and convergence (1.6) were analyzed in [8], the essential ingredient is the so-called "contractivity" properties of the interpolation operator Π_1 and Π_2, namely,

$$\|\nabla(\Pi_2 v - v)\|_{L^2(M)} \leq q_1 \|\nabla v\|_{L^2(M)} \text{ for any } v \in V_{1,h}, M \in \mathcal{M}_h,$$

$$\|\nabla(\Pi_1 v - v)\|_{L^2(M)} \leq q_2 \|\nabla v\|_{L^2(M)} \text{ for any } v \in V_{2,h}, M \in \mathcal{M}_h,$$

for some positive constants $q_i < 1, i = 1, 2$. For rectangular meshes in two dimensions, such "contractivity" properties were proved in [20]. However, such properties do not hold for rectangular meshes in three dimensions, see [20]. Hence, the analysis of the well-posedness of problem (1.3) and convergence of algorithm (1.4) on rectangles in three dimensions is missing in literature so far.

The purpose of the paper is to develop a new framework to analyze the well-posedness of problem (1.3) and the convergence in 2D and the monotone decreasing property in 3D of