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INTEGRABLE DISCRETISATION OF THE LOTKA-VOLTERRA SYSTEM*

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Abstract

In this paper, we apply Hirota's discretisation to a three-dimensional integrable Lotka-Volterra system. By analyzing the three-dimensional modified equation of the resulting numerical method, we show that it is volume-preserving, and has two independent first integrals. Moreover, it can be formally reduced to a system in one dimension via a volume-preserving transformation. If the given initial value is located in the positive octant, we prove that the numerical solution is confined to a one-dimensional connected and compact space which is diffeomorphic to a circle.

Mathematics subject classification: 65L12, 65P99, 37M05.
Key words: Integrable Lotka-Volterra system, Hirota's integrable discretisation, Backward error analysis, Modified differential equation.

1. Introduction

Integrable systems play an important role in applied fields like classical mechanics, fluid dynamics and quantum physics, revealing various rigid phenomena such as the quasi-periodic motion of celestial bodies, solitons in shallow water waves, etc. Many physical systems can be taken as perturbations of integrable systems. Generally speaking, a system is integrable if it can be solved by quadrature. In classical mechanics, the best known integrability is complete integrability, which was proposed by Liouville in the 19th century based on the notion of first integrals (conserved quantities). A Hamiltonian system is said to be integrable in Liouville's sense if it has sufficiently many independent first integrals in involution. This notion of integrability is applicable to systems of PDEs and also to discrete systems such as systems on

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lattices. Compared with non-integrable systems, integrable systems have much more specific properties, and show regular motions and more predictable long-term behavior. For example, the flow of a completely integrable Hamiltonian system turns out to be linear flows on invariant tori in suitable coordinates. Although some integrable systems can be solved analytically by the inverse scattering transform or inverse spectral methods etc., most of them could not be treated successfully in this way. Therefore, in the study of integrable systems solving the system numerically provides an alternative approach.

Geometric numerical integration is a class of numerical methods in the spirit of preserving the intrinsic properties of the system [10]. This idea has been successfully applied to construct symplectic (Poisson) methods for Hamiltonian (Poisson) systems, volume-preserving methods for source-free systems, and integral-preserving methods for systems with given first integrals. It has been confirmed theoretically that geometric numerical integration usually provides numerical results with superior qualitative behavior compared to other methods. For integrable Hamiltonian systems, it has been proved in [2, 20, 21] that symplectic methods can preserve most of the invariant tori of the systems, with torus deformations of magnitude compatible with the accuracy of the method, as long as the time-step size falls in a Cantor set of relatively large measure near the origin of the real line. On each of the tori the numerical orbits turn out to be quasi-periodic with diophantine frequencies, and therefore are ergodic and densely fill the invariant torus. For completely integrable Hamiltonian systems, symplectic methods can generate bounded numerical solutions with linear error growth near the preserved invariant tori over exponentially long time intervals [2, 5, 21].

Our purpose of this paper is to understand integrable discretisations and to find clues for designing proper numerical methods for integrable systems. The integrable discretizations have been studied extensively since the mid 70s of the last century. In the construction of integrable discretizations, the most powerful technique is the bilinear approach proposed by Hirota using the bilinear formalism to guarantee the integrability of the discretisation [6, 7]. In [13], a scheme following from Hirota's discretisation was given and a significant comparison was made with a symplectic integrator for the Lotka-Volterra system. Kahan's method [8] preserves the integrability of certain quadratic systems, including the three-dimensional Lotka-Volterra system. The geometric properties of this discretisation are analyzed in [1]. We investigate Hirota's integrable discretisation for the pendulum problem and the Lotka-Volterra system, and give numerical methods which preserve the phase volume and the first integrals of the system. By means of backward error analysis [5], we interpolate the numerical solution of the integrable systems into exact solutions of the corresponding modified differential equations (MDEs), with the modified vector fields written in a formal power series in terms of h (time-step). Moreover, we prove that the MDEs of the numerical method derived from Hirota's discretisation inherit most of the properties of the original systems. The outline of this paper is as follows. In Section 2, we give the definitions of integrable systems and integrable discretisations. In Section 3, we analyze the numerical solutions of the method derived from Hirota's integrable discretisation for the two-dimensional pendulum problem. In Section 4, we introduce the Lotka-Volterra (LV) system and analyze its geometric properties. We discretize the LV system by Hirota's discretisation and establish the corresponding MDEs in Section 5. As the modified vector field of the Hirota scheme is a polynomial for the LV system, it is shown that the MDEs can be calculated by a recurrence formula in terms of the coefficients of the polynomial. Section 6 is devoted to reformulating the MDEs by using its geometric properties. Some related analyses are also presented in this section. We end this paper with concluding remarks.