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THE PERFORMANCE OF ORTHOGONAL MULTI-MATCHING PURSUIT UNDER THE RESTRICTED ISOMETRY PROPERTY *

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Abstract

The orthogonal multi-matching pursuit (OMMP) is a natural extension of the orthogonal matching pursuit (OMP). We denote the OMMP with the parameter M as OMMP(M) where $M \geq 1$ is an integer. The main difference between OMP and OMMP(M) is that OMMP(M) selects M atoms per iteration, while OMP only adds one atom to the optimal atom set. In this paper, we study the performance of orthogonal multi-matching pursuit under RIP. In particular, we show that, when the measurement matrix A satisfies $(25s, 1/10)$-RIP, OMMP(M_0) with $M_0 = 12$ can recover s-sparse signals within s iterations. We furthermore prove that OMMP(M) can recover s-sparse signals within $O(s/M)$ iterations for a large class of M.

Mathematics subject classification: 94A12, 65H99, 65D15.
Key words: Sparse signals, Compressed sensing, Greedy algorithms

1. Introduction

1.1. Orthogonal Matching Pursuit

Orthogonal matching pursuit (OMP) is a popular algorithm for the recovery of sparse signals and it is also commonly used in compressed sensing. Let A be a matrix of size $m \times N$ and $\mathbf{y}$ be a vector of size m. The aim of OMP is to find an approximate solution to the following ℓ_0-minimization problem:

$$\min_{\mathbf{x}\in\mathbb{C}^N} \|\mathbf{x}\|_0 \qquad \text{s.t.} \qquad A\mathbf{x} = \mathbf{y},$$

where $\|\mathbf{x}\|_0$ denotes the number of non-zero entries in $\mathbf{x}$. In compressed sensing and the sparse representation of signals, we often have $m \ll N$. Throughout this paper, we suppose that the columns of the sampling matrix $A \in \mathbb{C}^{m\times N}$ are ℓ_2-normalized.

To introduce the performance of OMP, we first recall the definition of the restricted isometry property (RIP) [2] which is frequently used in the analysis of the recovering algorithm in compressed sensing. We say that the signal $\mathbf{x}$ is *s-sparse* if $\|\mathbf{x}\|_0 \leq s$ and use Σ_s to denote the set of s-sparse signals, i.e.,

$$\Sigma_s = \{\mathbf{x} \in \mathbb{C}^N : \|\mathbf{x}\|_0 \leq s\}.$$

Following Candès and Tao, for $1 \leq s \leq N$ and $\delta_s \in [0, 1)$, we say that the matrix A satisfies (s, δ_s)-RIP if

$$(1-\delta_s)\|\mathbf{x}\|_2^2 \leq \|A\mathbf{x}\|_2^2 \leq (1+\delta_s)\|\mathbf{x}\|_2^2 \tag{1.1}$$

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holds for all s-sparse signals $\mathbf{x}$. When there is no confusion, we may omit the subscript s in the notation δ_s.

Theoretical analysis of OMP has concentrated primarily on two directions. The first one is to study the condition for the matrix A under which OMP can recover s-sparse signals in exactly s iterations. In this direction, one uses the coherence and RIP to analyze the performance of OMP. In particular, Davenport and Wakin showed that, when the matrix A satisfies $(s+1, \frac{1}{3\sqrt{s}})$-RIP, OMP can recover s-sparse signal in exactly s iterations [4]. The sufficient condition is improved to $(s+1, \frac{1}{\sqrt{s}+1})$-RIP in [8, 9] (see also [6, 7, 13]). However, it was observed in [12], when the matrix A satisfies $(c_0 s, \delta_{c_0 s})$-RIP for some fixed constants $c_0 > 1$ and $0 < \delta_{c_0 s} < 1$, that s iterations of OMP is not enough to uniformly recover s-sparse signals, which implies that OMP has to run for more than s iterations to uniformly recover the s-sparse signals. Hence, one investigates the performance of OMP along the second line with allowing to OMP run more than s iterations. For this case, it is possible that OMP add wrong atoms to the optimal atom set, but one can identify the correct atoms by the least square. A main result in this direction is presented by Zhang [15] with proving that when A satisfies $(31s, 1/3)$-RIP OMP can recover the s-sparse signal in at most $30s$ iterations.

The other type of greedy algorithms, which are based on OMP, have been proposed including the regularized orthogonal matching pursuit (ROMP) [10], subspace pursuit (SP) [3], CoSaMP [11], and many other variants. For each of these algorithms, it has been shown that, under a natural RIP setting, they can recover the s-sparse signals within $O(s)$ iterations.

1.2. Orthogonal Multi-matching Pursuit and Main Results

A more natural extension of OMP is the orthogonal multi-matching pursuit (OMMP) [7]. We denote the OMMP with the parameter M as OMMP(M) where M is an integer. Throughout this paper, we assume that $M \in [1, s]$. The main difference between OMP and OMMP(M) is that OMMP(M) selects M atoms per iteration, while OMP only adds one atom to the optimal atom set. The Algorithm 1 outlines the procedure of OMMP(M) with initial feature set Λ^0. In comparision with OMP, OMMP has fewer iterations and computational complexity [6]. We note that, when $M = 1$, OMMP(M) is identical to OMP. OMMP is also studied in [6, 8, 14] under the names of KOMP, MOMP and gOMP, respectively. These results show that, when RIP constant $\delta = O(\sqrt{M/s})$, OMMP(M) can recover the s-sparse signal in s iterations.

The aim of this paper is to study the performance of OMMP(M) under a more natural setting of RIP (the RIP constant is an absolute constant). Particularly, we also would like to understand the relation between the number of iterations and the parameter M. So, we are interested in the following questions:

Question 1 *Does there exist an absolute constant M_0 so that* OMMP(M_0) *can recover all the s-sparse signals within s iterations?*

Question 2 *For $1 \leq M \leq s$, can* OMMP(M) *recover the s-sparse signals within $O(s/M)$ iterations?*

We try to answer the two questions for a general case where the measurement vector $\mathbf{y}$ is corrupted by noise $\mathbf{e} \in \mathbb{C}^m$, i.e., $\mathbf{y} = A\mathbf{x} + \mathbf{e}$. We next state one of our main results which gives an affirmative answer to Question 1. To state conveniently, throughout the rest of this paper, we assume that C is a constant only depending on the RIP constant of the matrix A.