Journal of Computational Mathematics
Vol.33, No.5, 2015, 533–556.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1506-m2014-0186

FULL-DISCRETE FINITE ELEMENT METHOD FOR STOCHASTIC HYPERBOLIC EQUATION*

Xiaoyuan Yang and Xiaocui Li
Department of Mathematics, Beihang University,
LMIB of the Ministry of Education, Beijing 100191, China
Email: xiaoyuanyang@vip.163.com, anny9702@126.com
Ruisheng Qi
Department of Mathematics, Northeastern University at Qinhuangdao, China
Email: qiruisheng123@sohu.com
Yinghan Zhang
Department of Mathematics, Beihang University,
LMIB of the Ministry of Education, Beijing 100191, China
Email: zhangyinghan007@126.com

Abstract

This paper is concerned with the finite element method for the stochastic wave equation and the stochastic elastic equation driven by space-time white noise. For simplicity, we rewrite the two types of stochastic hyperbolic equations into a unified form. We convert the stochastic hyperbolic equation into a regularized equation by discretizing the white noise and then consider the full-discrete finite element method for the regularized equation. We derive the modeling error by using "Green's method" and the finite element approximation error by using the error estimates of the deterministic equation. Some numerical examples are presented to verify the theoretical results.

Mathematics subject classification: 60H15, 65M60.
Key words: Stochastic hyperbolic equation; Strong convergence; Additive noise; Wiener process.

1. Introduction

Nowadays, stochastic partial differential equations (SPDEs) are accepted as being a very suitable framework to understand complex phenomenon. Hence, various numerical methods and approximation schemes for SPDEs have been developed, analyzed, and tested, see, e.g., [1, 11, 16, 21, 24, 27–33]. From a computational view-point, SPDEs are usually handled by using finite difference methods [1, 14] and finite element methods [5, 6, 16]. However, finite element methods for SPDEs may provide a more flexible framework than finite difference methods and allow for space (or space-time) adaptively.

Finite element methods for stochastic parabolic equations (SPEs) developed in [1] have been the starting point of several investigations. In [27], the semigroup framework is firstly applied to linear stochastic parabolic equation driven by white noise and then strong convergence is obtained by using the error estimates of the corresponding deterministic equation. Similar estimates for some nonlinear stochastic parabolic equation are considered in [3, 22, 28]. However,

* Received January 13, 2015 / Revised version received May 15, 2015 / Accepted June 25, 2015 / Published online September 18, 2015 /

both for spatial and temporal approximation, the obtained order of convergence is of the suboptimal form. Based on optimal spatial and temporal results in [15,19], optimal error estimates for spatially semidiscrete and for full-discrete approximation scheme are obtained in [20] . In [21], a new full-discrete approximation of SPDEs is presented based on using a standard finite element method for the spatial discetization and a stochastic exponential integrator scheme for the temporal discretization. Compared with the above schemes, the results are more general since the linear operator doesn't need to be self-adjoint. The semigroup framework is also used in the stochastic hyperbolic equation (see [16]). In order to study the finite element method, in [16], the authors write the stochastic wave problem as an abstract equation. In fact, to our best knowledge, there are very few results about finite element method for stochastic partial differential equation driven by space-time Brownian sheet noise [13], especially for stochastic hyperbolic equation. For more contributions about finite element methods for SPDEs, we refer to [12, 13, 17, 22].

Here we consider the following equation

$$\frac{\partial X(t)}{\partial t} + \mathcal{A}X(t) = \mathrm{B}\dot{W}(t,x), \quad t \in [0,T],\ X(0) = X_0, \tag{1.1}$$

where X_0 is a function defined on $\mathcal{D} = [0, L]$, $\mathcal{A}$ is a linear operator and W is space-time white noise defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

In section 2, we give a precise information about $\mathcal{A}$ and B. Let $S(t)$ be the semigroup generated by the operator $\mathcal{A}$ and $\mathrm{G}(t; x, y)$ be the Green's function of $S(t)$. Then the mild solution of the problem in the Green's function framework is given by

$$X(t,x) = \int_{\mathrm{D}} \mathrm{G}(t;x,y)X_0(y)\mathrm{d}y + \int_0^t \int_{\mathcal{D}} \mathrm{G}(t-s;x,y)W(\mathrm{d}s,\mathrm{d}y). \tag{1.2}$$

In [13], strong error estimates for stochastic linear fourth-order parabolic equation driven by space-time Brownian sheet noise are proved in the Green's function framework. In this paper, we will follow the idea of [13] to study finite element method for two types of stochastic hyperbolic equation including stochastic wave equation and stochastic elastic equation. In order to approximate the problem, we introduce a regularized equation by discretizing the space-time white noise. Then a modeling error is obtained based on representation of the exact and regularized equation's solutions by Green's functions. Next the full-discrete finite element approximations to the solution of the regularized problem are obtained by using, for discretization in space, a standard Galerkin finite element method and, for time-stepping, the Backward Euler method. In order to obtain error estimates of the finite element approximation, we give the discrete Green's function of discrete operator and the error estimates for the corresponding deterministic equation.

The paper is organized as follows. In Section 2, we give some basis notations and then present two types of stochastic hyperbolic equation. In Section 3, we introduce a regularized stochastic equation by discretizing the space-time white noise and study the modeling error. In Section 4 and 5, we study stochastic exponential integrator scheme and the full-discrete finite element method for the regularized equation. Finally, some numerical experiments are given in Section 6 to verify our theoretical results.