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INEXACT TWO-GRID METHODS FOR EIGENVALUE PROBLEMS*

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Abstract

We discuss the inexact two-grid methods for solving eigenvalue problems, including both partial differential and integral equations. Instead of solving the linear system exactly in both traditional two-grid and accelerated two-grid method, we point out that it is enough to apply an inexact solver to the fine grid problems, which will cut down the computational cost. Different stopping criteria for both methods are developed for keeping the optimality of the resulting solution. Numerical examples are provided to verify our theoretical analyses.

Mathematics subject classification: 65N25, 65N30, 65B99.
Key words: Inexact, Two-grid, Eigenvalue, Eigenvector, Finite element method, Convergence rate.

1. Introduction

The purpose of this paper is to present inexact two-grid methods for solving eigenvalue problems, including both partial differential and integral equations.

The research on two-grid method was initialized by Xu [1–3] for nonselfadjoint and indefinite problems, and has been extensively developed. Then, it was applied to other problems by many other authors, for instance, Axelsson and Layton [4] for nonlinear elliptic problems, Dawson and Wheeler [5] for nonlinear parabolic equations, Layton and Lenferink [6], Utnes [7], and Layton and Tobiska [8] for Navier-Stokes problems, Marion and Xu [9] for evolution problems. This method is also used as part of the finite difference scheme (see also Dawson, Wheeler and Woodward [10] for parabolic equations). Recently, Chien and Jeng [11] used this method along with the continuation method for solving semilinear elliptic eigenvalue problems; Jin, Shu and Xu [12] employed it for decoupling systems of partial differential equations; Xu and Zhou [13,14] developed localized and parallelized algorithms based on two-grid discretizations for linear and nonlinear elliptic boundary problems as well as eigenvalue problems.

Two-grid method is first applied to eigenvalue problems by Xu and Zhou [15]. The correction step is similar to the early work in 1981 by Lin and Xie [16] in which both nonlinear and eigenvalue problems were discussed. Later, they also developed the localized and parallelized version [17]. Based on this two-grid idea, Dai and Zhou [18] presented a three scale methods for the eigenvalue problems in quantum mechanics. Hu and Cheng [19] raised an acceleration technique for two-grid method. Xie and Lin [20] generalized the two-grid where the sizes of h and H are independent with each other.

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In this paper, we present an inexact two-grid algorithms for eigenvalue problems: Find $\lambda_h \in \mathbb{R}$ and $u_h \in \mathcal{S}_h \setminus \{0\}$ such that

$$a(u_h, v) = \lambda_h b(u_h, v), \qquad \forall v \in \mathcal{S}_h.$$

Here $\mathcal{S}_h$ is a finite element space defined on a quasi-uniform grid of size h. Our inexact two-grid algorithms for eigenvalue problem reads

1. Solve a standard finite element discretization on a coarse space $\mathcal{S}_H$: Find $\lambda_H \in \mathbb{R}$ and $u_H \in \mathcal{S}_H \setminus \{0\}$ such that

$$a(u_H, v) = \lambda_H b(u_H, v), \qquad \forall v \in \mathcal{S}_H. \tag{1.1}$$

obtaining an initial guess for the eigenpair on fine grid space $\mathcal{S}_h$.

2. Choose and solve a linear system inexactly on fine grid space depending on the mesh parameters.

   - If $h \geq H^2$, we adopt the *inexact two-grid scheme* (ITG). Find $u^h \in \mathcal{S}_h$ such that

$$a(u^h, v) = b(u_H + \xi d, v) \tag{1.2}$$

   - If $h < H^2$, we employ the *inexact accelerated two-grid scheme* (IATG). Find $u^h \in \mathcal{S}_h$ such that

$$a(u^h, v) - \lambda_H b(u^h, v) = b(u_H + \xi d, v), \tag{1.3}$$

   Here d is the b-normalized function that is proportional to the residual, and ξ is the b-norm of the residual. Note that in the second strategy, a shifted linear system is solved, which is also discussed in [21].

3. Recover the eigenvalue by

$$\lambda^h = \frac{a(u^h, u^h)}{b(u^h, u^h)}. \tag{1.4}$$

Our main contribution is in the second step, *i.e.*, the fine grid correction. Specifically, instead of solving the linear system exactly, which might be expensive to compute or even not available, we resort to inexact solvers to get a decent solution for the current discretization level. The accuracy of the solution and the computational cost of the total algorithm depends on the stopping criteria of the inexact solver. In comparison with the result in Xu and Zhou [15] where the optimal grid parameters are set as $h = H^2$, by the analysis of Section 3, we obtain that

- ITG with the stopping criteria $\tau = \mathcal{O}(H^2)$ on relative residual norm,
- IATG with the stopping criteria $\tau = \mathcal{O}(1) < 1$ on relative residual norm,

we both have

$$\|u^h - u\|_a \leq \mathcal{O}(h^r + H^{r+1}) \quad \text{and} \quad \left|\lambda^h - \lambda\right| \leq \mathcal{O}(h^{2r} + H^{2r+2}). \tag{1.5}$$

Both methods reduce the computational cost comparing to Xu and Zhou's method while ITG is slightly cheaper than IATG. In comparison with the result of Hu and Cheng [19] in which