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ON THE NUMERICAL SOLUTION OF ELLIPTIC AND PARABOLIC PDE IN THE REAL PROJECTIVE PLANE*

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Abstract

Some models dealing with fibers and liquid crystals can be formulated probabilistically in terms of orientation distributions. Since the orientation of a thin object can be specified by a point in a real projective plane this approach leads to elliptic and parabolic problems in the real projective plane. In most previous works these kind of problems have been considered on the unit sphere which is a double cover of the real projective plane. However, numerically this is inefficient because the resulting systems of equations are unnecessarily big. We formulate the problem directly in the real projective plane using a certain parametrization with three coordinate domains. After reducing the computations to the coordinate domains we can then use finite elements almost in a standard way. In particular the standard error estimates with usual Sobolev spaces remain valid in this setting. We consider both elliptic and parabolic cases, and demonstrate the validity of our approach.

Mathematics subject classification: 65M60, 35Q84, 35Q35, 58J32
Key words: Smoluchowski equation, Real projective plane, Finite elements, Orientation distribution.

1. Introduction

In this article we are interested in computing the solutions to elliptic and parabolic problems in the real projective plane using finite element methods. These kind of problems arise in various applications: dilute suspension of wood fibres [7, 19, 21], liquid crystals [8, 17] and even the analysis of images [4]. In all cases the unknown function is a probability distribution of the orientation of the fibers or liquid crystals or other objects.[1)] Recall that the orientation of a rigid body can be specified by a point in $\mathbb{SO}(3)$ which is diffeomorphic to the real projective space $\mathbb{RP}^3$. Now in the applications described in the references cited above one may assume that the object is in fact a very thin rod so it is reasonable to ignore its rotation around its long axis. Hence the orientation can be specified by a point in the real projective plane $\mathbb{RP}^2$.

In all previous publications that we are aware of the relevant equations have been analyzed on the unit sphere S^2 instead of $\mathbb{RP}^2$. This is possible since S^2 is the 2 sheeted covering space of $\mathbb{RP}^2$. Hence one obtains correct results if one assumes that all relevant functions satisfy $f(p) = f(-p)$ and vector fields satisfy $w(p) = -w(-p)$. Anyway it would appear more natural to do the analysis directly on $\mathbb{RP}^2$.

There has also been some interest in the numerical solution of the models discussed above, see for example [2,5,6,11,12] and references therein. In these numerical studies the problem has

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1) The relevant equation is sometimes called Smoluchowski equation or Fokker-Planck equation.

also been analyzed in S^2. Numerically this is not anymore equivalent to doing the computations in $\mathbb{RP}^2$. In the discretization there are in this case twice as many unknowns than necessary, and hence the numerical cost of solving the corresponding linear system is typically 4 to 8 times more expensive. In particular in time dependent problems where a linear system must be solved at each time step, the amount of unnecessary computation can be quite substantial.

In previous numerical studies typically S^2 has been regarded as a submanifold of $\mathbb{R}^3$. Hence one part of the discretiztion error is that one approximates S^2 by some other structure. This approach would be somewhat involved with $\mathbb{RP}^2$ because it cannot in any case be embedded in $\mathbb{R}^3$. However, such an embedding is unnecessary and we will below show how to introduce some convenient parametrizations of $\mathbb{RP}^2$ which allows us to do computations directly on $\mathbb{RP}^2$.

Note also that S^2 and $\mathbb{RP}^2$ cannot be parametrized using a single coordinate patch. For example using the spherical coordinates, like in [2], to parametrize S^2 creates artificial singularities at the poles. While it is possible to mitigate the effects of these singularities by appropriate numerical tricks this nevertheless has an adverse influence on the stability and accuracy of the computations. In our approach we cover $\mathbb{RP}^2$ with 3 coordinate patches and hence there are no singularities due to parametrization.

The content of the article is as follows. In Section 2 we recall some background material form Riemannian geometry and PDE theory. In Section 3 we describe the discretization of the problem which reduces the computations to the standard finite element setting. In Sections 4 and 5 we present the numerical results in some elliptic and parabolic test cases and finally in section 6 we give some conclusions and perspectives for future work.

2. Preliminaries

We start by recalling some facts from the theory of PDEs and differential geometry. More details can be found in [9, 13, 14, 18, 23].

2.1. Differential geometry

Let g be a Riemannian metric on some smooth manifold M. The components of g in the coordinate system are denoted by g_{ij}, and the resulting matrix is G. The components of G^{-1} are denoted by g^{ij}. Let M be a Riemannian manifold with boundary ∂M. The canonical volume form of M is denoted by ω_M and the volume of M is thus $\mathsf{vol}(M) = \int_M \omega_M$. The induced volume form on ∂M is denoted by $\omega_{\partial M}$. In case of nonorientable manifolds ω_M is interpreted as Riemannian density.

Then if w is some vector field on M we can define the gradient of u by the formula

$$g(\mathsf{grad}(u), w) = du\, w = w(u).$$

The standard gradient (resp. divergence and Laplacian) operator in Euclidean spaces is denoted by ∇ (resp. $\nabla\cdot$ and Δ) as usual. In a coordinate system we can write

$$\mathsf{grad}(u) = G^{-1}\nabla u = \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_i},$$
$$g(\mathsf{grad}(u),\ \mathsf{grad}(v)) = \langle \nabla u, G^{-1}\nabla v\rangle.$$