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STRONG PREDICTOR-CORRECTOR METHODS FOR STOCHASTIC PANTOGRAPH EQUATIONS*

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Abstract

The paper introduces a new class of numerical schemes for the approximate solutions of stochastic pantograph equations. As an effective technique to implement implicit stochastic methods, strong predictor-corrector methods (PCMs) are designed to handle scenario simulation of solutions of stochastic pantograph equations. It is proved that the PCMs are strong convergent with order $\frac{1}{2}$. Linear MS-stability of stochastic pantograph equations and the PCMs are researched in the paper. Sufficient conditions of MS-unstability of s-tochastic pantograph equations and MS-stability of the PCMs are obtained, respectively. Numerical experiments demonstrate these theoretical results.

Mathematics subject classification: 60H10, 65C20.
Key words: Stochastic pantograph equation, Predictor-corrector method, MS-convergence, MS-stability.

1. Introduction

In 1971, Ockendon and Tayler [15] used the equation $\dot{x(t)} = ax(t) + bx(pt)$ to model the collection of current by pantograph of an electric locomotive. This is the origin of the 'pantograph' in 'pantograph differential equations'. From then on, pantograph differential equations arise widely in dynamical systems, probability, quantum mechanics, electrodynamics and so on. A wealth of literature exists on analytical solution as well as numerical solution. The early related results can be found in [5, 8, 10], and the referents therein. More recently results can be found in [2, 4, 9, 12, 13].

Stochastic pantograph equation can be viewed as a generalization of the deterministic pantograph differential equation which takes into account of random factors. It possesses a wide range of applications. Up to now, only few results of stochastic pantograph equation have been presented. In 2000, Baker and Buckwar [1] obtained the necessary analytical theory for existence and uniqueness of strong approximations of a continuous extension of the θ-Euler methods and established 1/2 mean-square convergence of approximations. In 2007, Fan, Liu and Cao [7] discussed the existence and uniqueness of solutions and convergence of semi-implicit euler methods for stochastic pantograph equation. Some criteria for linear asymptotically mean square stability was given in [6]. In 2009, Xiao and Zhang [17] proved θ-methods of nonlinear stochastic pantograph equation are MS-stabile under appropriate conditions. In 2011, Xiao

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and Zhang [19]constructed numerical methods with variable step size to solve stochastic pantograph equation, the convergence and linear MS-stability were discussed. In 2013, Xiao,Zhang and Qin [18] discusses the MS-stability of the milstein method for stochastic pantograph equations.

For deterministic ordinary differential equations, the numerical stability of explicit methods are generally worse than implicit methods and the disadvantage of implicit methods is that an algebraic equation needs to be solved at each time step. It has been well known that PCMs can improve numerical stability comparing with standard explicit methods and don't require to solve an algebraic equations. For SDEs, the PCMs have the same properties. Weak PCMs for SDEs were discussed in [16] and [11]. In [3], a family of strong predictor-corrector Euler methods is designed to simulate the solution of SDEs. In [14], Niu and Zhang established a class of PCMs for SDEs and proved that the PCMs maintain almost sure and moment exponential stability for sufficiently small timesteps.As far as we know, it doesn't exist any literature about PCMs for SDEs with delay (SDDEs).

In this article, we deal with stochastic pantograph equation

$$\begin{cases} dx(t) = f(x(t), \quad x(pt))dt + g(x(t), \quad x(pt))dw(t), \quad t_0 < t, \quad p \in (0,1); \\ x(t) \quad = \xi(t), \quad pt_0 \le t \le t_0. \end{cases} \tag{1.1}$$

The work is organized as follows: Section 2 analyzes mean-square bound and stability of stochastic pantograph differential equations and establishes a family of predictor-corrector methods (PCMs (θ, η)) to simulate approximation of the stochastic pantograph differential equations. In Section 3, the convergence is discussed. It proved that the PCMs (θ, η) is mean square numerical convergent with order 1/2. In Section 4, some linear numerical MS-stability criteria of PCMs (θ, η) are obtained. If stochastic pantograph differential equations are MS-stable, then the numerical solutions of PCMs (θ, η) are MS-stable under appropriate conditions. Section 5 gives some numerical experiments to illustrate the obtained theoretical results.

2. Predictor-corrector Methods for Stochastic Pantograph Differential Equations

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a filtration $(\mathcal{A}_t)_{t \ge t_0}$, which is right-continuous and satisfies that each $\mathcal{A}_t$ $(t \ge t_0)$ contains all P-null sets in $\mathcal{A}$, and w is an d-dimensional Brownian motion defined on the probability space, $|\cdot|$ is the trace norm, $E_t(\cdot) = E(\cdot|\mathcal{A}_t)$.

We integrate(1.1) and obtain

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), \quad x(ps))ds + \int_{t_0}^{t} g(x(s), \quad x(ps))dw(s), \quad t_0 \le t \le T, \\ x(t) = \xi(t), \quad pt_0 \le t \le t_0, \end{cases} \tag{2.1}$$

where $x(t)$ is a R^d-value random process, $p \in (0,1)$ denotes a given constant, the second integral is Itô type, $f : R^d \times R^d \to R^d$ and $g : R^d \times R^d \to R^d$ are two given Borel-measurable functions, $\xi(t)$ is a $C([pt_0, t_0], R^d)$-value initial segment with $E\|\xi(t)\| < \infty$, let $\|\xi\| = \sup_{pt_0 \le t \le t_0} |\xi(t)|$. Throughout this paper, we assume the Eq. (2.1) has an uniqueness solution $x(t) \in \mathcal{M}^2(R_+, R^d)$ and satisfies the following conditions:

(C1) (global Lipschitz condition)

$$|f(x,u) - f(y,v,)|^2 \vee |g(x,u) - g(y,v)|^2 \le \beta_1(|x-y|^2 + |u-v|^2); \tag{2.2}$$

(C2) (linear growth condition)

$$|f(x,u)|^2 \leq \beta_2(|x|^2+|u|^2); \quad |g(x,u)|^2 \leq \beta_3(|x|^2+|u|^2). \tag{2.3}$$

For convenience of discussion, let $\beta = \max\{\beta_2, \beta_3\}$, then, following lemma is obtained.

Lemma 2.1. *The solution of* (2.1) *which satisfy* C1 *and* C2 *has the property*

$$E\left(\sup_{pt_0\leq t\leq T}|x(t)|^2\right) \leq c_1, \tag{2.4}$$

moreover, for any $pt_0 \leq s < t \leq T$,

$$E|x(t)-x(s)|^2 \leq c_2(t-s), \tag{2.5}$$

here, c_1, c_2 *are nonnegative constants which don't depend on* s *and* t.

Proof. For $t_0 \leq t \leq T$, according to inequality $(a+b+c)^2 \leq 3a^2+3b^2+3c^2$, Hölder inequality and Burkhölder-Davis-Gundy inequality, it follows that

$$\begin{aligned} & E(\sup_{t_0\leq r\leq t}|x(r)|^2) \\ \leq\ & 3E|x(t_0)|^2 + 3(T-t_0)\int_{t_0}^t E|f(x(r),x(pr))|^2dr + 12\int_{t_0}^t E|g(x(r),x(pr))|^2dr. \end{aligned}$$

By linear growth condition C2, we have

$$\begin{aligned} & E(\sup_{t_0\leq r\leq t}|x(r)|^2) \\ \leq\ & 3E|x(t_0)|^2 + 3\beta(T-t_0+4)(\int_{t_0}^t E|x(r)|^2dr + \int_{t_0}^t E|x(pr)|^2dr) \\ \leq\ & 3E|x(t_0)|^2 + 3\beta(T-t_0+4)(1+\tfrac{1}{p})\int_{pt_0}^t E|x(r)|^2dr \\ \leq\ & 3E|x(t_0)|^2 + 3\beta(T-t_0+4)(1+\tfrac{1}{p})\int_{pt_0}^t E(\sup_{pt_0\leq r\leq t}|x(r)|^2)dr. \end{aligned}$$

So,

$$\begin{aligned} & E(\sup_{pt_0\leq r\leq t}|x(r)|^2) \\ \leq\ & E(\sup_{pt_0\leq r\leq t_0}|\xi(r)|^2) + E(\sup_{t_0\leq r\leq t}|x(r)|^2) \\ \leq\ & 4E(\sup_{pt_0\leq r\leq t_0}|\xi(r)|^2) + 3\beta(T-t_0+4)(1+\tfrac{1}{p})\int_{pt_0}^t E(\sup_{pt_0\leq r\leq t}|x(r)|^2)dr. \end{aligned}$$

(2.4) is obtained from Gronwall's inequality.

Using Hölder's inequality, it follows that

$$\begin{aligned} & E|x(t)-x(s)|^2 \\ =\ & E|\int_s^t f(x(r),x(pr))dr + \int_s^t g(x(r),x(pr))dw(r)|^2 \\ \leq\ & 2(t-s)\int_s^t E|f(x(r),x(pr))|^2dr + 2\int_s^t E|g(x(r),x(pr))|^2dr \\ \leq\ & 2(T-pt_0+1)\beta\int_s^t E[|x(r)|^2+|x(pr)|^2]dr. \end{aligned} \tag{2.6}$$

Combining (2.6) with (2.4), (2.5) holds. This completes the proof. □

We consider stochastic predictor-corrector methods with predictor process

$$\bar{x}_{n+1} = x_n + f(x_n, \hat{x}_n)h + g(x_n, \hat{x}_n) \bigtriangleup w_n, \tag{2.7}$$

and corrector process

$$x_{n+1} = x_n + h[\theta f(\bar{x}_{n+1}, \hat{\bar{x}}_{n+1}) + (1-\theta) f(x_n, \hat{x}_n)] \\ +[\eta g(\bar{x}_{n+1}, \hat{\bar{x}}_{n+1}) + (1-\eta) g(x_n, \hat{x}_n)] \bigtriangleup w_n, \tag{2.8}$$

where, $t_n = t_0 + nh$, x_n, $\hat{x}_n$ and $\hat{\bar{x}}_{n+1}$ are approximations to $x(t_n)$, $x(pt_n)$ and $x(pt_{n+1})$, respectively. $x_n = \xi(t_n)$ when $n \le 0$ and $\bigtriangleup w_n = w(t_{n+1}) - w(t_n)$. When $pt_n > t_0$, there exists an integer $v_n (0 \le v_n < n)$ and $\delta_n \in [0, 1)$ such that $pt_n = t_0 + v_n h + \delta_n h$. Let

$$\hat{x}_n = \begin{cases} \delta_n x_{v_n+1} + (1-\delta_n) x_{v_n}, & pt_n > t_0, \quad n = 0, 1, \cdots, \\ \xi(pt_n), & pt_n \le t_0, \end{cases} \tag{2.9}$$

$$\hat{\bar{x}}_{n+1} = \begin{cases} x_{v_{n+1}}, & pt_{n+1} > t_0, \quad n = 0, 1, \cdots, \\ \xi(pt_{n+1}), & pt_{n+1} \le t_0. \end{cases} \tag{2.10}$$

We denote the methods (2.7)-(2.10) as $PCMs(\theta, \eta)$.

3. Convergence Analysis

Lemma 3.1. ([7]) *EM-methods for stochastic pantograph equations is convergent with order* $\frac{1}{2}$.

Theorem 3.1. *If f,g satisfy* C1 *and* C2*, then* $PCMs(\theta, \eta)$ *converge with order* $\frac{1}{2}$.

Proof. It follows from Lemma3.1 that

$$\begin{aligned} E_{t_0}|x_{n+1} - x(t_{n+1})|^2 &\le 2E_{t_0}|x_{n+1} - \bar{x}_{n+1}|^2 + 2E_{t_0}|\bar{x}_{n+1} - x(t_{n+1})|^2 \\ &= 2E_{t_0}|x_{n+1} - \bar{x}_{n+1}|^2 + O(h). \end{aligned} \tag{3.1}$$

By condition C1, we have

$$\begin{aligned} &E_{t_0}|x_{n+1} - \bar{x}_{n+1}|^2 \\ &\le 2h^2\theta^2 E_{t_0}|f(\bar{x}_{n+1}, \hat{\bar{x}}_{n+1}) - f(x_n, \hat{x}_n)|^2 + 2\eta^2 h E_{t_0}|g(\bar{x}_{n+1}, \hat{\bar{x}}_{n+1}) - g(x_n, \hat{x}_n)|^2 \\ &\le 2\beta_1 h(h\theta^2 + \eta^2)[E_{t_0}|\bar{x}_{n+1} - x_n|^2 + E_{t_0}|\hat{\bar{x}}_{n+1} - \hat{x}_n|^2]. \end{aligned} \tag{3.2}$$

Combining linear growth condition C2 with (2.4) yields

$$\begin{aligned} &E_{t_0}|\bar{x}_{n+1} - x_n|^2 \\ &\le 2\beta h(1+h)[E_{t_0}|x_n|^2 + E_{t_0}|\hat{x}_n|^2] \\ &\le 2\beta h(1+h)[2E_{t_0}|x_n - x(t_n)|^2 + 2E_{t_0}|x(t_n)|^2 + 2E_{t_0}|\hat{x}_n - x(pt_n)|^2 + 2E_{t_0}|x(pt_n)|^2] \\ &\le 4\beta h(1+h)[E_{t_0}|x_n - x(t_n)|^2 + E_{t_0}|\hat{x}_n - x(pt_n)|^2] + O(h). \end{aligned} \tag{3.3}$$

By Lemma 2.1, we have

$$E_{t_0}|\hat{\bar{x}}_{n+1} - \hat{x}_n|^2 \le 3E_{t_0}|\hat{\bar{x}}_{n+1} - x(pt_{n+1})|^2 + 3E_{t_0}|x(pt_n) - \hat{x}_n|^2 + O(h). \tag{3.4}$$

Here,

$$\begin{aligned} &E_{t_0}|x(pt_n) - \hat{x}_n|^2 \\ &\le 4\delta_n^2 E_{t_0}|x_{v_n+1} - x(t_{v_n+1})|^2 + 4(1-\delta_n)^2 E_{t_0}|x_{v_n} - x(t_{v_n})|^2 \\ &\quad +4\delta_n^2 E_{t_0}|x(t_{v_n+1}) - x(pt_n)|^2 + 4(1-\delta_n)^2 E_{t_0}|x(t_{v_n}) - x(pt_n)|^2 \\ &\le 4E_{t_0}|x_{v_n+1} - x(t_{v_n+1})|^2 + 4E_{t_0}|x_{v_n} - x(t_{v_n})|^2 + O(h), \end{aligned} \tag{3.5}$$

and

$$
\begin{aligned}
&E_{t_0}|\hat{\bar{x}}_{n+1} - x(pt_{n+1})|^2 \\
&\leq 2E_{t_0}|x_{v_{n+1}} - x(t_{v_{n+1}})|^2 + 2E_{t_0}|x(t_{v_{n+1}}) - x(t_{v_{n+1}} + \delta_{n+1}h)|^2 \\
&\leq 2E_{t_0}|x_{v_{n+1}} - x(t_{v_{n+1}})|^2 + O(h).
\end{aligned} \tag{3.6}
$$

Substituting (3.1) with (3.2)-(3.6) and an induction lead to

$$E_{t_0}|x_{n+1} - x(t_{n+1})|^2 = O(h).$$

This completes the proof. □

4. Linear Stability of $PCMs(\theta, \eta)$

For convenience, writing $PCMs(\theta, \eta)$ as

$$
\begin{cases}
x_{n+1} = x_n + hF_\theta^n + G_\eta^n \triangle w_n, \\
F_\theta^n = \theta f(\bar{x}_{n+1}, \hat{\bar{x}}_{n+1}) + (1-\theta) f(x_n, \hat{x}_n), \\
G_\eta^n = \eta g(\bar{x}_{n+1}, \hat{\bar{x}}_{n+1}) + (1-\eta) g(x_n, \hat{x}_n).
\end{cases} \tag{4.1}
$$

Considering linear scalar pantograph differential equation

$$
\begin{cases}
dx(t) = [ax(t) + bx(pt)]dt + [\rho x(t) + \sigma x(pt)]dw(t), \quad t \geq 0, \\
x(0) = \xi,
\end{cases} \tag{4.2}
$$

where $a, b, \rho, \sigma \in R, p \in (0,1)$ and $\xi \neq 0$ with probability 1.

Theorem 4.1. (1) *The solution of* (4.2) *is MS-stable whenever*

$$2a + 2|b + \rho\sigma| + \rho^2 + \sigma^2 < 0. \tag{4.3}$$

(2) *The solution of* (4.2) *is MS-unstable whenever*

$$
\begin{cases}
2a + \rho^2 > 0, \\
2a\sigma^2 - b^2 - 2b\rho\sigma \geq 0.
\end{cases} \tag{4.4}
$$

Proof. (1) cf. [6]; (2) By Itô formula, it yields

$$
\begin{aligned}
d|x(t)|^2 &= [2x(t)(ax(t) + bx(pt)) + |\rho x(t) + \sigma x(pt)|^2]dt \\
&\quad + 2x(t)(\rho x(t) + \sigma x(pt))dw(t) \\
&= [(2a + \rho^2)|x(t)|^2 + 2(b + \rho\sigma)x(t)x(pt) + \sigma^2|x(pt)|^2]dt \\
&\quad + [2\rho|x(t)|^2 + 2\sigma x(t)x(pt)]dw(t).
\end{aligned} \tag{4.5}
$$

Taking expectations after both sides integral from 0 to t, we have

$$E|x(t)|^2 = E|\xi|^2 + E\int_0^t [(2a + \rho^2)|x(s)|^2 + 2(b + \rho\sigma)x(s)x(ps) + \sigma^2|x(ps)|^2]ds. \tag{4.6}$$

Assuming $2a + \rho^2 > 0$ follows that

$$
\begin{aligned}
E|x(t)|^2 = &\; E|\xi|^2 + E\int_0^t \left(\sqrt{2a + \rho^2}|x(s)| + \tfrac{b + \rho\sigma}{\sqrt{2a + \rho^2}}|x(ps)|\right)^2 ds \\
&+ \left(\sigma^2 - \tfrac{(b + \rho\sigma)^2}{2a + \rho^2}\right) E\int_0^t |x(ps)|^2 ds.
\end{aligned} \tag{4.7}
$$

So, when $2a+\rho^2>0$ and $2a\sigma^2-b^2-2b\rho\sigma\geq 0$, it follows that

$$E|x(t)|^2 \geq E|\xi|^2. \tag{4.8}$$

This means the Eq. (4.2) is MS-unstable. This completes the proof. □

Theorem 4.2. *PCMs (θ,η) with stepsize h for linear stochastic pantograph Eq. (4.2) is MS-stable, if*

$$\begin{aligned}&R(\bar a,\bar b,\bar\rho,\bar\sigma,\theta,\eta)\\ =\;&\bar a+\bar a^2+\tfrac{3-\theta}{2}|\bar b|+(2-\theta)\bar b^2+\bar\rho+(2-\eta)\bar\sigma+(3-\tfrac{3\theta}{2}+2\theta|\bar b|)|\bar a\bar b|\\ &+(3-\tfrac{3\eta}{2}+2\eta|\bar b|)\sqrt{\bar\rho\bar\sigma}+(\theta\bar a^2+\eta\bar\rho)(|1+\bar a|+\tfrac{3}{2}\bar b)^2\\ &+(\theta\bar a^2+3\eta\bar\rho)(\sqrt{\bar\rho}+\tfrac{3}{2}\sqrt{\bar\sigma})^2+(\theta|\bar a\bar b|+\eta\sqrt{\bar\rho\bar\sigma})(2|1+\bar a|+|\bar\sigma|)<0,\end{aligned} \tag{4.9}$$

where $\bar a=ah,\bar b=bh,\bar\rho=\rho^2h,\bar\sigma=\sigma^2h$.

Proof. Applying the methods(4.1) to (4.2), it follows that

$$\begin{aligned}E|x_{n+1}|^2 &= E|x_n+hF_\theta^n+G_\eta^n\triangle w_n|^2\\ &\leq E|x_n|^2+2(h^2E|F_\theta^n|^2+E|G_\eta^n\triangle w_n|^2+hEx_nF_\theta^n+Ex_nG_\eta^n\triangle w_n).\end{aligned} \tag{4.10}$$

By $E\triangle w_n=E\triangle w_n^3=0, E\triangle w_n^2=h, E\triangle w_n^4=3h^2$, we have

$$\begin{aligned}&Ex_nF_\theta^n\\ &=Ex_n[\theta(a\bar x_{n+1}+b\hat{\bar x}_{n+1})+(1-\theta)(ax_n+b\hat x_n)]\\ &\leq (a+h\theta a^2+\tfrac{\theta h|ab|+|b|}{2})Ex_n^2+\tfrac{\theta h|ab|+(1-\theta)|b|}{2}E\hat x_n^2+\tfrac{\theta|b|}{2}E\hat{\bar x}_{n+1}^2,\end{aligned} \tag{4.11}$$

$$\begin{aligned}&Ex_nG_\eta^n\triangle w_n\\ &=Ex_n[\eta(\rho\bar x_{n+1}+\sigma\hat{\bar x}_{n+1})+(1-\eta)(\rho x_n+\sigma\hat x_n)]\triangle w_n\\ &=\eta\rho^2hEx_n^2+\eta\rho\sigma hEx_n\hat x_n\\ &\leq(\eta\rho^2h+\tfrac{\eta h|\rho\sigma|}{2})Ex_n^2+\tfrac{\eta h|\rho\sigma|}{2}E\hat x_n^2,\end{aligned} \tag{4.12}$$

$$\begin{aligned}&E|F_\theta^n|^2\\ &\leq\theta E|(a\bar x_{n+1}+b\hat{\bar x}_{n+1})|^2+(1-\theta)E|(ax_n+b\hat x_n)|^2\\ &=\{\theta a^2[(1+ha)^2+\rho^2h+|1+ha||hb|+|\rho\sigma|h]+\theta|ab||1+ah|\\ &\quad+(1-\theta)(a^2+|ab|)\}Ex_n^2+\{\theta a^2(h^2b^2+\sigma^2h+|1+ha||hb|+|\rho\sigma|h)+\theta|a|b^2h\\ &\quad+(1-\theta)(b^2+|ab|)\}E\hat x_n^2+\theta\{b^2+|ab||1+ah|+|a|b^2h\}E\hat{\bar x}_{n+1}^2,\end{aligned} \tag{4.13}$$

$$\begin{aligned}&E|G_\eta^n\triangle w_n|^2\\ &\leq\eta E|(\rho\bar x_{n+1}+\sigma\hat{\bar x}_{n+1})\triangle w_n|^2+(1-\eta)E|(\rho x_n+\sigma\hat x_n)\triangle w_n|^2\\ &\leq\{\eta\rho^2h[(1+ha)^2+3\rho^2h+|1+ha||hb|+3|\rho\sigma|h]+\eta|\rho\sigma|h|1+ah|\\ &\quad+(1-\eta)(\rho^2h+|\rho\sigma|h\}Ex_n^2\\ &\quad+\{\eta\rho^2h(h^2b^2+3\sigma^2h+|1+ha||hb|+3|\rho\sigma|h)+\eta|b\rho\sigma|h^2+(1-\eta)(\sigma^2h+|\rho\sigma|h)\}E\hat x_n^2\\ &\quad+\eta\{\sigma^2h+|\rho\sigma h||1+ah|+|b\rho\sigma|h^2\}E\hat{\bar x}_{n+1}^2.\end{aligned} \tag{4.14}$$

Combining (4.10) with (4.11)-(4.14), yields

$$\begin{aligned}&E|x_{n+1}|^2\\ &\leq E|x_n|^2+2R_1(\bar a,\bar b,\bar\rho,\bar\sigma,\theta,\eta)Ex_n^2+2R_2(\bar a,\bar b,\bar\rho,\bar\sigma,\theta,\eta)E\hat x_n^2+2R_3(\bar a,\bar b,\bar\rho,\bar\sigma,\theta,\eta)E\hat{\bar x}_{n+1}^2,\end{aligned}$$

where

$$
\begin{aligned}
R_1 &= \bar{a} + \theta\bar{a}^2 + \tfrac{\theta|\bar{a}\bar{b}|+|\bar{b}|}{2} + \theta[\bar{a}^2(1+\bar{a})^2 + \bar{a}^2\bar{\rho}] + \theta\bar{a}^2(|1+\bar{a}||\bar{b}| + \sqrt{\bar{\rho}\bar{\sigma}}) \\
&\quad + \theta|\bar{a}\bar{b}||1+\bar{a}| + (1-\theta)\bar{a}^2 + (1-\theta)|\bar{a}\bar{b}| + \eta\bar{\rho} + \tfrac{\eta\sqrt{\bar{\rho}\bar{\sigma}}}{2} \\
&\quad + \eta[\bar{\rho}(1+\bar{a})^2 + 3\bar{\rho}^2] + \eta\bar{\rho}(|1+\bar{a}||\bar{b}| + 3\sqrt{\bar{\rho}\bar{\sigma}}) + \eta\sqrt{\bar{\rho}\bar{\sigma}}|1+\bar{a}| + (1-\eta)\bar{\rho} + (1-\eta)\sqrt{\bar{\rho}\bar{\sigma}} \\
&= \bar{a} + \bar{a}^2 + \bar{\rho} + (1-\tfrac{\theta}{2})|\bar{a}\bar{b}| + \tfrac{1}{2}|\bar{b}| + (\theta\bar{a}^2 + \eta\bar{\rho})[(1+\bar{a})^2 + |1+\bar{a}||\bar{b}|] + (\theta\bar{a}^2 + 3\eta\bar{\rho})[\bar{\rho} + \sqrt{\bar{\rho}\bar{\sigma}}] \\
&\quad + (\theta|\bar{a}\bar{b}| + \eta\sqrt{\bar{\rho}\bar{\sigma}})|1+\bar{a}| + (1-\tfrac{\eta}{2})\sqrt{\bar{\rho}\bar{\sigma}},
\end{aligned}
$$

$$
\begin{aligned}
R_2 &= \tfrac{\theta|\bar{a}\bar{b}|}{2} + \tfrac{(1-\theta)|\bar{b}|}{2} + \theta\bar{a}^2(\bar{b}^2 + \bar{\sigma}) + \theta\bar{a}^2(|1+\bar{a}||\bar{b}| + \sqrt{\bar{\rho}\bar{\sigma}}) \\
&\quad + \theta|\bar{a}|\bar{b}^2 + (1-\theta)\bar{b}^2 + (1-\theta)|\bar{a}\bar{b}| + \tfrac{\eta\sqrt{\bar{\rho}\bar{\sigma}}}{2} \\
&\quad + \eta\bar{\rho}(\bar{b}^2 + 3\bar{\sigma}) + \eta\bar{\rho}(|1+\bar{a}||\bar{b}| + 3\sqrt{\bar{\rho}\bar{\sigma}}) + \eta|\bar{b}|\sqrt{\bar{\rho}\bar{\sigma}} + (1-\eta)\bar{\sigma} + (1-\eta)\sqrt{\bar{\rho}\bar{\sigma}} \\
&= (1-\tfrac{\theta}{2})|\bar{a}\bar{b}| + (\theta\bar{a}^2 + \eta\bar{\rho})[\bar{b}^2 + |1+\bar{a}||\bar{b}|] + \theta|\bar{a}|\bar{b}^2 + (\theta\bar{a}^2 + 3\eta\bar{\rho})[\bar{\sigma} + \sqrt{\bar{\rho}\bar{\sigma}}] \\
&\quad + (1-\theta)(\tfrac{|\bar{b}|}{2} + \bar{b}^2) + \eta|\bar{b}|\sqrt{\bar{\rho}\bar{\sigma}} + (1-\tfrac{\eta}{2})\sqrt{\bar{\rho}\bar{\sigma}} + (1-\eta)\bar{\sigma},
\end{aligned}
$$

$$
R_3 = \theta(\tfrac{|\bar{b}|}{2} + \bar{b}^2) + (\theta|\bar{a}\bar{b}| + \eta\sqrt{\bar{\rho}\bar{\sigma}})(|1+\bar{a}| + |\bar{b}|) + \eta\bar{\sigma}.
$$

By (2.9) and (2.10), considering $\delta_n \in [0,1)$, it holds

$$
E|x_{n+1}|^2 \le E|x_n|^2 + 2\Big(R_1 E|x_n|^2 + R_2(E|x_{v_n+1}|^2 + E|x_{v_n}|^2) + R_3 E|x_{v_{n+1}}|^2\Big).
$$

We have

$$
R_1 + 2R_2 + R_3 \le R.
$$

Thus,

$$
E|x_{n+1}|^2 \le \Big(1 + 2R(\bar{a}, \bar{b}, \bar{\rho}, \bar{\sigma}, \theta, \eta)\Big)\max\{E|x_n|^2, E|x_{v_n+1}|^2, E|x_{v_n}|^2, E|x_{v_{n+1}}|^2\}.
$$

It means that the method is MS-stable when $R < 0$. □

Theorem 4.3. *If*

$$
a + \frac{3-\theta}{2}|b| + (3 + \frac{\eta}{2})|\rho\sigma| + (1+\eta)\rho^2 + (2-\eta)\sigma^2 < 0,
$$

then there exists an $h_0(a, b, \rho, \sigma, \theta, \eta) \in (0,1)$ such that PCMs(θ, η) with stepsize $h \in (0, h_0]$ for Eq. (4.2) are MS-stable.

Proof. Without loss of generality, assume that $h < 1$. A combining of (4.10)-(4.14) leads to

$$
\begin{aligned}
E|x_{n+1}|^2 \le \ & \Big\{1 + [a + \frac{3-\theta}{2}|b| + (3 - \frac{\eta}{2})|\rho\sigma| + (1+\eta)\rho^2 + (2-\eta)\sigma^2]h \\
& + H(a, b, \rho, \sigma, \theta, \eta)h^2\Big\}\max\{E|x_n|^2, E|x_{v_n+1}|^2, E|x_{v_n}|^2, E|x_{v_{n+1}}|^2\},
\end{aligned}
$$

where

$$
\begin{aligned}
& H(a, b, \rho, \sigma, \theta, \eta) \\
= \ & (1+\theta)a^2 + (2-\theta)b^2 + (3 + \tfrac{1}{2}\theta + 2\theta|b|)|ab| + (\theta a^2 + \eta\rho^2)\Big(2|a| + a^2 + 3|b||1+a| + b^2\Big) \\
& + (\theta a^2 + 3\eta\rho^2)(\rho^2 + 3|\rho\sigma| + \sigma^2) + 2\eta|b|\sqrt{\rho\sigma} + (\theta|ab| + \eta|\rho\sigma|)(2|a| + |b|).
\end{aligned}
\tag{4.15}
$$

So, take

$$h_0 = \min\left\{ \frac{-[a + \frac{3-\theta}{2}|b| + (3 + \frac{\eta}{2})|\rho\sigma| + (1+\eta)\rho^2 + (2-\eta)\sigma^2]}{H(a,b,\rho,\sigma,\theta,\eta)}, 1 \right\}. \tag{4.16}$$

Then when stepsize $h \in (0, h_0]$, the methods PCMs (θ, η) are MS-stable. □

Remark 4.1. The stepsize restriction of stability is obtained by repeated use of inequality $2ab \le a^2 + b^2$, so PCMs (θ, η) have better stability region than which is determined by inequality $R < 0$. From Section 5, one can see that the numerical approximations of PCMs with stepsize $h > h_0$ which make $R > 0$ may still remain MS-stable. How to obtain an exact critical stepsize $\tilde{h}_0$ such that the method is stable for $h \in (0, \tilde{h}_0]$ and unstable for $h \in (\tilde{h}_0, +\infty)$? This is an interesting topic. So far, we have not found any research which has obtained sufficient and necessary bounds for numerical methods for SDDEs. We will focus on it in the future work.

5. Numerical Illustration

In the section, we give some numerical examples to illustrate the obtained theoretical results. Consider the following stochastic pantograph equations:

Example 5.1. *Consider the linear stochastic pantograph equation*

$$\begin{cases} dx(t) = \Big(ax(t) + bx(\frac{t}{2})\Big)dt + \Big(\rho x(t) + \sigma x(\frac{t}{2})\Big)dw(t), \\ x(0) = 2. \end{cases} \tag{5.1}$$

According to (4.3), the equations with $(a, b, \rho, \sigma) = (-4, 1, 0.1, 1)$ and $(a, b, \rho, \sigma) = (-4, 0.1, 0, 1)$ are stable and the stability coefficient are -4.79 and -6.8, respectively. According to (4.4), the equations with $(a, b, \rho, \sigma) = (1, 1, 0.1, 1)$ and $(a, b, \rho, \sigma) = (1, 0.1, 0, 1)$ are unstable.

Example 5.2. *Nonlinear stochastic pantograph equation*

$$\begin{cases} dx(t) &= -\frac{1}{4}x(t)\Big(1 + cos^2 x(\frac{t}{2})\Big)dt + \frac{1}{5}x(t)x(\frac{t}{2})dw(t), \quad t > 0, \\ x(0) &= 2. \end{cases} \tag{5.2}$$

For numerical illustrating the convergency, we use the approximation formula

$$error = \sqrt{\frac{1}{10^4}\Sigma_{i=1}^{100}\Sigma_{j=1}^{100}|x(\omega_{i,j}, T) - x_N^{(i,j)}|^2}.$$

Table 5.1: Errors of PCMs (θ, η) for (5.1) with $(a, b, \rho, \sigma) = (-4, 1, 0.1, 1)$.

(θ, η)	$h = 2^{-5}$	$h = 2^{-6}$	$h = 2^{-7}$	$h = 2^{-8}$	$h = 2^{-9}$
$(0, 0)$	0.0094	0.0062	0.0041	0.0030	0.0021
$(0, 1/2)$	0.0093	0.0062	0.0042	0.0030	0.0022
$(0, 1)$	0.0107	0.0072	0.0050	0.0035	0.0025
$(1/2, 0)$	0.0089	0.0060	0.0041	0.0030	0.0021
$(1/2, 1/2)$	0.0087	0.0060	0.0041	0.0030	0.0022
$(1/2, 1)$	0.0100	0.0070	0.0049	0.0035	0.0025
$(1, 0)$	0.0093	0.0061	0.0041	0.0030	0.0021
$(1, 1/2)$	0.0091	0.0061	0.0042	0.0031	0.0022
$(1, 1)$	0.0103	0.0070	0.0049	0.0036	0.0025

Table 5.2: Errors of PCMs (θ, η) for (5.2).

(θ,η)	$h=2^{-5}$	$h=2^{-6}$	$h=2^{-7}$	$h=2^{-8}$	$h=2^{-9}$
$(0,0)$	0.4045e-003	0.3597e-003	0.2138e-003	0.1692e-003	0.0887e-003
$(0,1/2)$	0.7689e-003	0.4027e-003	0.3478e-003	0.2585e-003	0.1837e-003
$(0,1)$	0.1833e-003	0.1182e-003	0.0760e-003	0.0526e-003	0.0383e-003
$(1/2,0)$	0.1179e-003	0.0496e-003	0.0502e-003	0.0390e-003	0.0341e-003
$(1/2,1/2)$	0.6500e-003	0.4192e-003	0.3643e-003	0.2713e-003	0.1784e-003
$(1/2,1)$	0.3748e-003	0.2604e-003	0.1698e-003	0.1569 -003	0.0888e-003
$(1,0)$	0.7215e-003	0.4184e-003	0.4017e-003	0.2093e-003	0.1588e-003
$(1,1/2)$	0.1426e-003	0.0763e-003	0.0591e-003	0.0480e-003	0.0274e-003
$(1,1)$	0.3807e-003	0.2659e-003	0.1711e-003	0.1574e-003	0.0889e-003

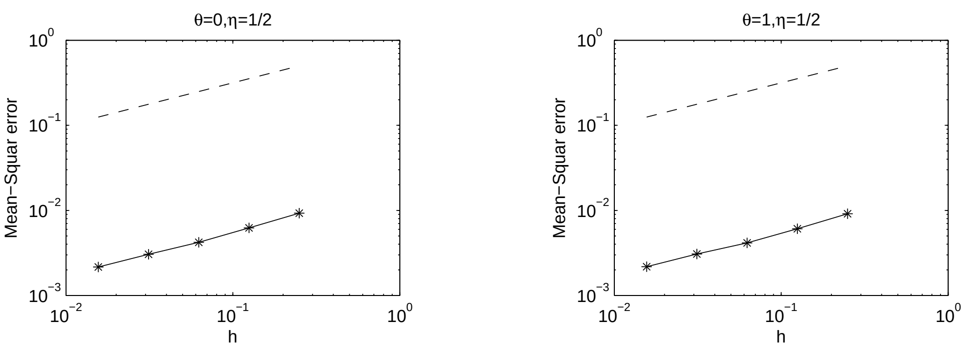
Table 5.3: Stability bound of (5.1).

$(a,b,\rho,\sigma)=(-4,1,0.1,1)$

(θ,η)	h_0	R
$(0,0)$	0.0063	1.3010e-018
$(0,1/2)$	0.0219	-0.0023
$(0,1)$	0.0372	-0.0080
$(1/2,0)$	0.0014	-5.6370e-004
$(1/2,1/2)$	0.0029	-0.0027
$(1/2,1)$	0.0044	-0.0065
$(1,0)$	0.0012	-7.6429e-004
$(1,1/2)$	0.0019	-0.0024
$(1,1)$	0.0027	-0.0048

$(a,b,\rho,\sigma)=(-4,0.1,0,1)$

(θ,η)	h_0	R
$(0,0)$	0.1074	-0.0566
$(0,1/2)$	0.1365	-0.0722
$(0,1)$	0.1655	-0.83267
$(1/2,0)$	0.0083	-0.0410
$(1/2,1/2)$	0.0106	-0.0232
$(1/2,1)$	0.0128	-0.0340
$(1,0)$	0.0044	-0.0080
$(1,1/2)$	0.0056	-0.0128
$(1,1)$	0.0067	-0.0187

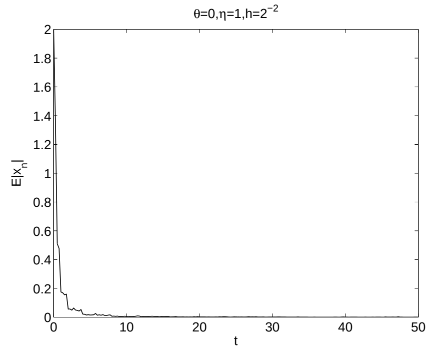


the errors figure of(5.1) $(a,b,\rho,\sigma)=(-4,1,0.1,1)$

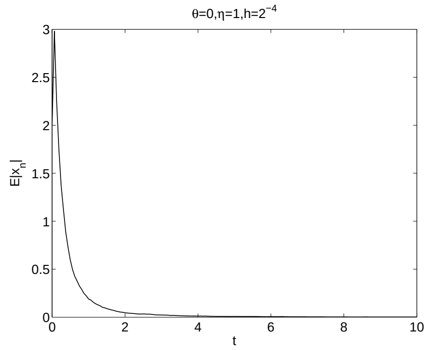
the errors figure of (5.2)

Fig. 5.1. Strong error plots

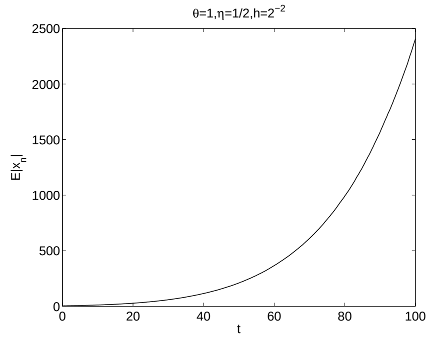
Considering the explicit analytical solutions of (5.1) and (5.2) are difficult to obtain, we take numerical solution with very small stepsize $h=2^{-13}$ as the exact solution. Applying the PCMs (θ,η) with stepsizes $2^q h$ $(q=4,5,6,7,8)$ to linear stochastic pantograph Eq. (5.1) and nonlinear stochastic pantograph Eq. (5.2) on interval [0,10], respectively, we can obtain errors (see Table 5.1 and 5.2). Two double-logarithmic graphs are constructed in Fig. 5.1, the slope of dashed



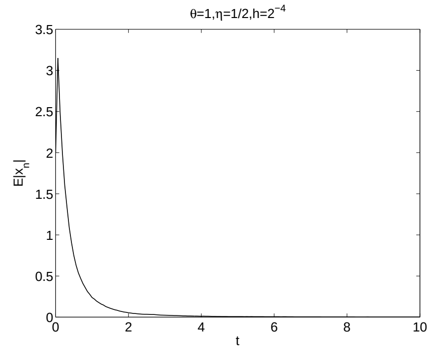
$(\theta, \eta) = (0, 1), h = 2^{-2}$



$(\theta, \eta) = (0, 1), h = 2^{-4}$

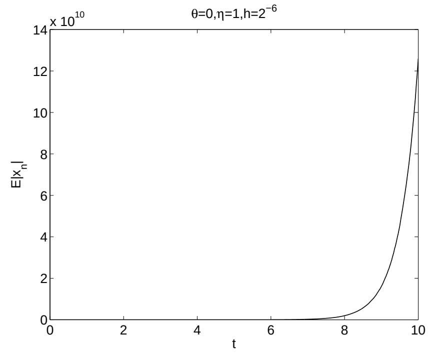


$(\theta, \eta) = (1, 1/2), h = 2^{-2}$

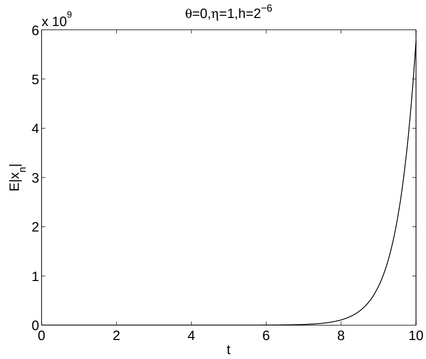


$(\theta, \eta) = (1, 1/2), h = 2^{-4}$

Fig. 5.2. the numerical solutions of (5.1) with $(a, b, \rho, \sigma) = (-4, 1, 0.1, 1)$.



$(a, b, \rho, \sigma) = (1, 1, 0.1, 1)$



$(a, b, \rho, \sigma) = (1, 0.1, 0, 1)$

Fig. 5.3. the numerical solutions of (5.1).

line is 1/2. We see that the slopes of the curves appear to match well, suggesting that the convergent order of the PCMs (θ, η) for (5.1) and (5.2) is 1/2.

For numerical illustrating the stability, we use the approximation formula

$$E(|x_n|) \approx \frac{1}{10^4} \Sigma_{i=1}^{10^4} |x_n(\omega_i)|.$$

In Table 5.3, stability bounds of PCMs (θ, η) for (5.1) are given. We see that taking appropriate parameters θ and η in PCMs (θ, η)can improve the stability of Euler-Maruyama method, where

h_0 and R are obtained by (4.16) and (4.9), respectively. According to Remark 4.4, the stablity regions of PCMs (θ, η) are larger than the data in Table 5.3. This can be demonstrated in Fig. 5.2. In Fig. 5.3, we can see that for unstable equations, the numerical solutions are unstable, too.

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