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AN ADAPTIVE HYBRID STRESS TRANSITION QUADRILATERAL FINITE ELEMENT METHOD FOR LINEAR ELASTICITY*

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Abstract

In this paper, we discuss an adaptive hybrid stress finite element method on quadrilateral meshes for linear elasticity problems. To deal with hanging nodes arising in the adaptive mesh refinement, we propose new transition types of hybrid stress quadrilateral elements with 5 to 7 nodes. In particular, we derive a priori error estimation for the 5-node transition hybrid stress element to show that it is free from Poisson-locking, in the sense that the error bound in the a priori estimate is independent of the Lamé constant λ. We introduce, for quadrilateral meshes, refinement/coarsening algorithms, which do not require storing the refinement tree explicitly, and give an adaptive algorithm. Finally, we provide some numerical results.

Mathematics subject classification: 65N12, 65N15, 65N30.
Key words: Hybrid stress element, Transition element, Adaptive method, Quadrilateral mesh, Poisson-locking, Plane elasticity.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, with boundary $\Gamma = \Gamma_N \cup \Gamma_D$ and $\text{meas}(\Gamma_D) > 0$. Let $\mathbf{n}$ be the outward unit normal vector on Γ. The plane linear elasticity problem reads

$$\begin{cases} -\mathbf{div}\,\sigma = \mathbf{f} & \text{in } \Omega \\ \sigma = \mathbb{C}\varepsilon(\mathbf{u}) & \text{in } \Omega \\ \sigma \cdot \mathbf{n}|_{\Gamma_N} = \mathbf{g}, & \mathbf{u}|_{\Gamma_D} = 0 \end{cases} \tag{1.1}$$

where $\sigma \in \mathbb{R}^{2\times 2}_{\text{sym}}$ is the symmetric stress tensor, $\mathbf{u} \in \mathbb{R}^2$ the displacement field, $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla + \nabla^T)\mathbf{u}$ the strain tensor, $\mathbf{f} \in \mathbb{R}^2$ the body loading density, and $\mathbf{g} \in \mathbb{R}^2$ the surface traction. Here $\mathbb{C}$ denotes the elasticity modulus tensor with $\mathbb{C}\varepsilon(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda \text{div}(\mathbf{u})\mathbb{I}$ and $\mathbb{I}$ is the 2×2 identity tensor. The constants μ, λ are the Lamé parameters, given by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

for plane strain problems and by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-\nu)}$$

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for plane stress problems, where $0 < \nu < 0.5$ is the Poisson's ratio and E is the Young's modulus.

Hybrid stress finite element method (also called assumed stress hybrid finite element method), based on Hellinger–Reissner variational principle and pioneered by Pian [24], is known to be an efficient approach [25–27,35,36,38] to improve the performance of the standard 4-node compatible displacement quadrilateral (bilinear) element, which yields poor results for problems with bending and, for plane strain problems, at the nearly incompressible limit. In [26] Pian and Sumihara derived a robust 4-node hybrid stress quadrilateral element (abbr. PS) through a rational choice of stress terms. Xie and Zhou [35, 36] proposed accurate 4-node hybrid stress quadrilateral elements by optimizing stress modes with a so-called energy-compatibility condition [41]. Yu et al. [38] analyzed the methods and obtained uniform convergence and a posteriori error estimation [26,35]. It is worth noticing that the 4-node hybrid stress finite element method is of almost the same computational cost as the bilinear Q4 element due to the local elimination of stress parameters.

Adaptive mesh refinement (AMR) for the numerical solution of the PDEs is a standard tool in science and engineering to achieve better accuracy with minimum degrees of freedom. The typical structure in one iteration of adaptive algorithms consists of four steps:

$$\textbf{Solve} \longrightarrow \textbf{Estimate} \longrightarrow \textbf{Mark} \longrightarrow \textbf{Refine/Coarsen.}$$

AMR methods locally refine/coarsen meshes according to the estimated error distribution through repeating the above working loop comprised of finite element solution, error estimation, element (edge or patch) marking, and mesh refinement/coarsening until the error decreases to a prescribed level. Classical recursive bisection and coarsening algorithms [19,28,29] are widely used in adaptive algorithms (see, for example, ALBERTA [30] and deal.II [3]). These algorithms make use of a refinement tree data structure and subroutines to store/access the refinement history.

Chen and Zhang [10] proposed a non recursive refinement/coarsening algorithm for triangular meshes which does not require storing the bisection tree explicitly. They only store coordinates of vertices and connectivity of triangles which are the minimal information required to represent a mesh for standard finite element computation. In fact, they build the bisection tree structure implicitly into a special ordering of the triangles and simplify the implementation of adaptive mesh refinement and coarsening—thus provided an easy-access interface for the usage of mesh adaptation without much sacrifice in computing time. These algorithms have been extended to 3D later by Bartels and Schreier [4].

Refinement and coarsening for adaptive quadrilateral meshes are more difficult than the counterparts for triangular meshes. When a 4-node quadrilateral element is subdivided into four smaller elements, hanging nodes might appear on the element boundaries of its immediate neighborhoods. There are several different approaches to deal with the hanging nodes. The first approach is to constrain the mid-side node displacement of the transition element to be the average of the displacement at the two corner nodes of the same edge [23,33]. However, this way nullifies the accuracy enhancement effect of the mid-side nodes and constraint equations are computationally inefficient [1]. Borouchaki and Frey [5] presented a method to convert the triangular mesh into a quadrilateral mesh, by which one can use the adaptive triangular mesh generation method and then convert the mesh to a quadrilateral one. Schneiders [31] provided some template elements for local refinement to connect different layer patterns. This method would keep the conformity of mesh, but at the same time, could introduce distorted elements.