Journal of Computational Mathematics
Vol.34, No.4, 2016, 385–406.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1601-m4541

PLANE WAVES NUMERICAL STABILITY OF SOME EXPLICIT EXPONENTIAL METHODS FOR CUBIC SCHRÖDINGER EQUATION*

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Abstract

Numerical stability when integrating plane waves of cubic Schrödinger equation is thoroughly analysed for some explicit exponential methods. We center on the following second-order methods: Strang splitting and Lawson method based on a one-parameter family of 2-stage 2nd-order explicit Runge-Kutta methods. Regions of stability are plotted and numerical results are shown which corroborate the theoretical results. Besides, a technique is suggested to avoid the possible numerical instabilities which do not correspond to continuous ones.

Mathematics subject classification: 65M12, 65M15, 65M99.
Key words: Numerical stability, Exponential splitting Lawson methods, Projection onto invariant quantities, Plane waves, Schrödinger equation.

1. Introduction

Numerical stability of plane wave solutions of cubic Schrödinger equation has already been a subject of research in the literature. In the first place, that analysis has been done in [11] for the first-order Lie splitting method when the initial condition is just a small perturbation of a constant. Secondly, in [6] the study has been performed for two implicit methods (Besse and Fei) for the more general case of small perturbations of initial conditions which have the form $u_0(x) = ae^{ikx}$ ($a \in \mathbb{C}, k \in \mathbb{R}$.) In all previous cases, the analysis has been based on linear stability in the sense of ignoring terms which are quadratic on the perturbation. Afterwards, an analysis has been performed [8] using also modulated Fourier expansions for Strang splitting method. That type of analysis is theoretically valid for longer times although requires more restrictions on the parameters of integration. However, up to our knowledge, there is no numerical corroboration of the benefits of being more restrictive.

On the other hand, quite recently explicit exponential splitting and Runge-Kutta-based Lawson methods [2–5, 9] have been thoroughly developed and recommended for cubic Schrödinger equation. The former conserve two invariants (norm and momentum) while the latter do not. However, in [3–5], the conclusions are that, after projecting (very cheaply) on one of the invariants (norm), we are also projecting onto another invariant (momentum) for many solutions.

* Received June 18, 2014 / Revised version received December 4, 2015 / Accepted January 26, 2016 / Published online June 28, 2016 /

Besides, plane wave solutions are among those. In the comparison with splitting methods in terms of computational efficiency [5], high order of accuracy in time and space is in favour of projected Lawson methods, although exponential splitting methods are also many times a preferable tool.

The aim of this paper is to analyse and compare both type of methods with respect to its behaviour in terms of numerical stability when integrating plane wave solutions. For that, we will center for simplicity and, as a first stage, on second order methods: Strang splitting method [12] and Lawson methods based on explicit 2-stage Runge-Kutta methods. The latter is a one-parameter family of methods and all the analysis will be made in terms of that parameter (d). Besides, we will consider the unprojected and projected variants of these methods. In the numerical experiments, we have chosen pseudospectral discretization because it is very accurate for regular solutions and because it fits perfectly with the analysis in terms of the different frequencies which is done throughout the paper for the continuous in space problem.

We will see that Strang splitting method exactly integrates in time exact plane wave solutions (i.e without considering rounding errors). Nevertheless, Lawson methods do not. However, when projecting onto the norm, we achieve that the error not only in the momentum but also in the Hamiltonian vanishes for these solutions, which led us to believe that projected methods would behave better in terms of stability.

On the one hand, one conclusion in the paper is that the results are independent of the value of the frequency k of the unperturbed wave in contrast with what happens with Besse & Fei methods in [6]. On the other hand, in the comparison among Strang and Lawson methods, when $|\lambda||a|^2$ is small enough (λ being a real parameter in the equation), all methods behave in a similar manner. However, when $|\lambda||a|^2$ is bigger, Strang method behaves better than projected Lawson integrator and the last one better than the unprojected one.

In any case, we also suggest a filtering technique so as to try to avoid the numerical instabilities with all numerical methods when the continuous problem is stable.

The paper is structured as follows. Section 2 gives some preliminaries on the continuous problem and the considered numerical integrators. In Section 3, the behaviour of all considered methods when integrating the exact plane wave is justified. Besides, the precise results on the numerical stability with all methods when integrating a plane wave are stated. For the sake of clarity, the proofs have been relegated to an appendix. In Section 4, the different regions of stability are plotted for Strang and Lawson methods corresponding to $d = 1$. Finally, in Section 5 the numerical performance is shown for the different methods, different initial conditions and different time stepsizes and the technique to avoid instabilities is suggested.

2. Preliminaries

2.1. Continuous problem

We will consider the equation

$$u_t = iu_{xx} - i\lambda|u|^2u, \quad \lambda \in \mathbb{R}, \tag{2.1}$$

with periodic boundary conditions in a certain interval, which we will take as $[0, 2\pi]$ for the sake of simplicity. It is well known that this problem has as invariant quantities