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LOW RANK APPROXIMATION SOLUTION OF A CLASS OF GENERALIZED LYAPUNOV EQUATION*

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Abstract

In this paper, we consider the low rank approximation solution of a generalized Lyapunov equation which arises in the bilinear model reduction. By using the variation principle, the low rank approximation solution problem is transformed into an unconstrained optimization problem, and then we use the nonlinear conjugate gradient method with exact line search to solve the equivalent unconstrained optimization problem. Finally, some numerical examples are presented to illustrate the effectiveness of the proposed methods.

Mathematics subject classification: 15A24, 65F30, 93A15.
Key words: Generalized Lyapunov equation, Bilinear model reduction, Low rank approximation solution, Numerical method.

1. Introduction

Denoted by $R^{n\times n}$ be the set of $n\times n$ real matrices, $SR^{n\times n}$ be the set of $n\times n$ real symmetric matrices, $SR_{+}^{n\times n}$ be the set of $n\times n$ real symmetric positive definite matrices. We write $B>0$ $(B\geq 0)$ if the matrix B is positive definite (semidefinite). The symbol B^T stands for the transpose of the matrix B, and the symbol $\otimes$ stands for the Kronecker product. For the $n\times n$ matrix $B=(b_1,b_2,\cdots,b_n)=(b_{ij})$, $[B]_{ij}$ stands for the element of the ith row and jth column, that is, $[B]_{ij}=b_{ij}$, and $\mathrm{vec}(B)$ stands for a vector defined by $\mathrm{vec}(B)=(b_1^T,b_2^T,\cdots,b_n^T)^T$. The symbols $\mathrm{rank}(B)$ and $tr(B)$ stand for the rank and trace of the matrix B, respectively. We use $\lambda_1(B)$ and $\lambda_n(B)$ to denote the maximal and minimal eigenvalues of an $n\times n$ symmetric matrix B, respectively. We use $\|B\|_F$ to denote the Frobenius norm of a matrix B.

In this paper, we consider the low rank approximation solution of the generalized Lyapunov equation

$$AX+XA^T+\sum_{j=1}^{m}N_jXN_j^T+Q=0, \tag{1.1}$$

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where $A, N_1, N_2, \cdots, N_m \in R^{n\times n}$, Q is an $n \times n$ symmetric semidefinite matrix, and

$$I_n \otimes A + A \otimes I_n + \sum_{j=1}^{m} N_j \otimes N_j \in SR_+^{n^2\times n^2}$$

The low rank approximation solution of (1.1) arises in bilinear model reduction, which can be stated as follows (see [4,11,38] for more details). Consider the following bilinear control systems

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{m} N_j x(t) u_j(t) + Bu(t), \tag{1.2a}$$

$$y_t = Cx(t), \quad x(0) = x_0, \tag{1.2b}$$

where $A, N_j \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{k\times n}$. Let

$$P_1(t_1) = e^{At_1}B, \tag{1.3a}$$

$$P_i(t_1, t_2, \cdots, t_i) = e^{At_i}[N_1 P_{i-1}, \cdots, N_m P_{i-1}], \quad i = 2, 3, \cdots. \tag{1.3b}$$

Then the reachability Gramian

$$P = \sum_{i=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P_i P_i^T dt_1 \cdots dt_i$$

of (1.2) satisfies (1.1) with $Q = BB^T$.

In the last few years there has been a constantly increasing interest in developing effective numerical methods for the standard Lyapunov equation (i.e. Eq. (1.1) with $m = 0$). The numerical methods can be generally separated into two classes. The first class consists of direct method, such as the Bartels-Stewart method [3] and the Hammarling method [20]. The second class is the iterative method, such as Krylov subspace method [22], ADI method [26], matrix sign function method [6], Smith's method [34], block successive overrelaxation method [35] and the matrix splitting methods [14]. In particular, Bai [2] presented a HSS iterative method for solving large sparse continuous Sylvester equations with non-Hermitian and positive definite (semidefinite) matrices. Motivated by the classical conjugate direction method for Hermitian positive definite linear systems, Deng, Bai and Gao [7] constructed orthogonal direction methods for solving two classes of linear matrix equations. Some other matrix equations were also studied in [8,9,15,29]. However, when $m > 0$, the theory and numerical methods for the generalized Lyapunov equation (1.1) are fewer than the case $m = 0$, due to the complicated structure. By means of the linear operator theory and spectral analysis, Damm [10] and Zhang-Chen [37] gave some sufficient conditions for the existence of a positive (semi)definite solution of Eq. (1.1), but how to verify these conditions is difficult. By using the vectorizing operator and Kronecker product, Huang [18,19] transformed Eq. (1.1) into a system of linear equations and derived some sufficient conditions for the existence of a symmetric solution. A parameter iterative method was constructed to compute the symmetric solution, but how to choose the optimal parameter is unknown.

Recent interests on the Lyapunov equation are directed more towards large and sparse coefficients matrices A, N_j and $Q = BB^T$ with very low rank, where B has only a few columns. In this case, the standard methods are often too expensive to be practical, and some low rank iterative methods become more viable choices. Common ones are based on the ADI or Smith method (see [12,25,29]), on Krylov subspace techniques (see [15,25,26]), and on low rank