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A WEAK GALERKIN MIXED FINITE ELEMENT METHOD FOR SECOND-ORDER ELLIPTIC EQUATIONS WITH ROBIN BOUNDARY CONDITIONS*

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Abstract

In this paper, we present a weak Galerkin (WG) mixed finite element method for solving the second-order elliptic equations with Robin boundary conditions. Stability and a priori error estimates for the coupled method are derived. We present the optimal order error estimate for the WG-MFEM approximations in a norm that is related to the L^2 for the flux and H^1 for the scalar function. Also an optimal order error estimate in L^2 is derived for the scalar approximation by using a duality argument. A series of numerical experiments is presented that verify our theoretical results.

Mathematics subject classification: 65N30, 65N15, 65N12, 74N20.
Key words: Second-order elliptic equations, Robin boundary conditions, Weak Galerkin, Weak divergence.

1. Introduction

The main goal of this paper is to introduce a weak Galerkin mixed finite element method for second-order elliptic equations with Robin boundary conditions. We consider the following problem which seeks a flux function $\boldsymbol{q}$ and a scalar function u defined in an open bounded polytopal domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$:

$$\alpha \boldsymbol{q} + \nabla u = 0, \qquad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{q} = f, \qquad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{q} \cdot \boldsymbol{n} - \beta u = g, \qquad \text{on } \partial\Omega, \tag{1.3}$$

where $\alpha = (\alpha_{i,j}(\boldsymbol{x}))_{d\times d} \in [L^\infty(\Omega)]^{d\times d}$ is a symmetric, positive definite matrix on the domain Ω. The variational formulation for the problem (1.1)-(1.3) is to seek $\boldsymbol{q} \in H(\text{div};\Omega)$ and $u \in L^2(\Omega)$ such that

$$(\alpha \boldsymbol{q}, \boldsymbol{p}) + \left\langle \beta^{-1} \boldsymbol{q} \cdot \boldsymbol{n}, \boldsymbol{p} \cdot \boldsymbol{n} \right\rangle_{\partial\Omega} - (u, \nabla \cdot \boldsymbol{p}) = \left\langle \beta^{-1} g, \boldsymbol{p} \cdot \boldsymbol{n} \right\rangle_{\partial\Omega}, \quad \forall \boldsymbol{p} \in H(\text{div};\Omega), \tag{1.4}$$

$$(\nabla \cdot \boldsymbol{q}, v) = (f, v), \quad \forall v \in L^2(\Omega), \tag{1.5}$$

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where $\nabla\cdot\boldsymbol{p}$ is the divergence of the vector-valued functions $\boldsymbol{p}$ on Ω, $(\cdot,\cdot)$ stands for the L^2-inner product in $L^2(\Omega)$, and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ is the inner product in $L^2(\partial\Omega)$.

The weak Galerkin method is a new numerical method which is designed by using a weakly defined differential operator instead of the classical one in the variational formulation. In [8], a WG method was first introduced and analyzed for second order elliptic equations based on a discrete weak gradient arising from local RT or BDM elements. After that lots of work on WG for solving PDEs are investigated [2–7, 17–22] . Due to the use of the RT and BDM elements, the WG finite element method of [8] was limited to classical finite element partitions of triangles/rectangles ($d = 2$) or tetrahedra/hexahedron ($d = 3$). The relative computational study was given in [4]. In [9], the idea of employing stabilization terms was introduced and the resulting WG method used the configuration of $\{P_k(T), P_k(e); [P_{k-1}(T)]^d\}$ which made the partitions no longer be limited to triangles/rectangles ($d = 2$) or tetrahedra/hexahedron ($d = 3$). In practice, allowing arbitrary shape in finite element partition provides a convenient flexibility in both numerical approximation and mesh generation, especially in regions where the domain geometry is complex. Such a flexibility is also very much appreciated in adaptive mesh refinement methods. In addition, [10] gave a new WG scheme using the configuration of $\{P_k(T), P_{k-1}(e); [P_{k-1}(T)]^d\}$ which reduces the degree of freedom. In [11], a weak Galerkin mixed finite element method was developed for the second order elliptic equation in the mixed form. All these methods mentioned above were introduced for Dirichlet problems. As for Neumann problems, we can find a detailed introduction in [12]. However, there is no corresponding error analysis.

The Robin condition arises naturally in heat conduction problems as well as in Physical Geodesy [15]- [16]. Robin problems are more general, so they are more widely used in practical problems. For this reason, we develop a weak Galerkin MFEM for the second order elliptic equations with Robin boundary conditions only in the mixed form, the primal form is obvious and direct, so it's omitted here. In this work, the essential difficulty is coming from the proof of inf-sup condition. To overcome it, we design a suitable norm and construct the function $\boldsymbol{v}$ that satisfies the inf-sup condition.

The rest of this paper is organized as follows. In Section 2, we introduce some standard notations. In Section 3, we present the basic definition of weak divergence operator and discrete weak divergence in some weakly-defined spaces. In Section 4, we define some local projection operators and present the WG-MFEM algorithm. In Section 5, we shall establish an optimal order error estimate for the WG-MFEM approximations in a norm that is related to the L^2 for the flux and H^1 for the scalar function. In Section 6, we derive an optimal order error estimate in L^2 for the scalar approximation by using a duality argument. We give some numerical examples to verify our theoretical results in Section 7. Finally, we present some technical estimates and some approximation properties which are useful in the convergence analysis as Appendix.

2. Some Notations

This section is devoted to developing function spaces that are used in the variational formulation of differential equations. We begin with an introduction to Lebesgue spaces, and we restrict our attention for simplicity to real-valued functions, f, on a given domain, D. For $1 \le p \le \infty$, we define the Lebesgue spaces

$$L^p(D) := \{f : \|f\|_{L^p(D)} < \infty\},$$