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THE IMPLICIT CONVEX FEASIBILITY PROBLEM AND ITS APPLICATION TO ADAPTIVE IMAGE DENOISING*

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Abstract

The implicit convex feasibility problem attempts to find a point in the intersection of a finite family of convex sets, some of which are not explicitly determined but may vary. We develop simultaneous and sequential projection methods capable of handling such problems and demonstrate their applicability to image denoising in a specific medical imaging situation. By allowing the variable sets to undergo scaling, shifting and rotation, this work generalizes previous results wherein the implicit convex feasibility problem was used for cooperative wireless sensor network positioning where sets are balls and their centers were implicit.

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1. Introduction

In this paper we are concerned with the following "implicit convex feasibility problem" (ICFP). Given set-valued mappings $C_s : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, $s = 1, 2, \ldots, S$, with closed and convex value sets, the ICFP is,

$$\text{Find a point } x^* \in \cap_{s=1}^{S} C_s(x^*). \tag{1.1}$$

We call the sets $C_s(x)$ "variable sets" for obvious reasons and include "implicit" in this problem name because the sets defining it are not given explicitly ahead of time. The problem is inspired by the work of Gholami et al. [21] on solving the cooperative wireless sensor network positioning problem in $\mathbb{R}^2$ ($\mathbb{R}^n$). There, the sets $C_s(x)$ are circles (balls) with varying centers. A special instance of the ICFP is obtained by taking fixed sets $C_s(x) \equiv C_s$, for all $x \in \mathbb{R}^n$, and all $s = 1, 2, \ldots, S$, yielding the well-known, see, e.g., [3], "convex feasibility problem" (CFP) which is,

$$\text{Find a point } x^* \in \cap_{s=1}^{S} C_s. \tag{1.2}$$

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The CFP formalism is at the core of the modeling of many inverse problems in various areas of mathematics and the physical sciences. This problem has been widely explored and researched in the last decades, see, e.g., [10, Section 1.3], and many iterative methods where proposed, in particular projection methods, see, e.g., [11]. These are iterative algorithms that use projections onto sets, relying on the principle that when a family of sets is present, then projections onto the given individual sets are easier to perform than projections onto other sets (intersections, image sets under some transformation, etc.) that are derived from the given individual sets.

Gholami et al. in [21] introduced the implicit convex feasibility problem (ICFP) in $\mathbb{R}^d$ ($d = 2$ or $d = 3$) into their study of the wireless sensor network (WSN) positioning problem. In their reformulation the variable sets are circles or balls whose centers represent the sensors' locations and their broadcasting range is represented as the radii. Some of these centers are known a priori while the rest are unknown and need to be determined. The WSN positioning problem is to find a point, in an appropriate product space, which represents the circles or balls centers. The precise relationship between the WSN problem and the ICFP can be found in [21, Section B]. For more details and other examples of geometric positioning problems, see [20, 22].

We focus on the ICFP in $\mathbb{R}^n$ and present projection methods for its solution. This expands and generalizes the special case treated in Gholami et al. [21]. Moreover, we demonstrate the applicability of our approach to the task of image denoising, where we impose constraints on the image intensity at every image pixel. Because the constraint sets depend on the unknown variables to be determined, the method is able to adapt to the image contents. This application demonstrates the usefulness of the ICFP approach to image processing.

The paper is structured as follows. In Section 2 we show how to calculate projections onto variable sets. In Section 3 we present two projection type algorithmic schemes for solving the ICFP, sequential and simultaneous, along with their convergence proofs. In Section 4 we present the ICFP application to image denoising together with numerical visualization of the performance of the methods. Finally, in Section 5 we discuss further research directions and propose a further generalization of the ICFP.

2. Projections onto Variable Convex Sets

We begin by recalling the split convex feasibility problem (SCFP) and the constrained multiple-set split convex feasibility problem (CMSSCFP) that will be useful to our subsequent analysis.

Problem 2.1. *Censor and Elfving* ([13]). *Given nonempty, closed and convex sets* $C \subseteq \mathbb{R}^n$, $Q \subseteq \mathbb{R}^m$ *and a linear operator* $T : \mathbb{R}^n \to \mathbb{R}^m$, *the* ***Split Convex Feasibility Problem*** *(SCFP) is:*

$$\text{Find a point } x^* \in C \text{ such that } T(x^*) \in Q. \tag{2.1}$$

Another related more general problem is the following.

Problem 2.2. *Masad and Reich* ([31]). *Let* $r, p \in \mathbb{N}$ *and* Ω_s, $1 \le s \le S$, *and* Q_r, $1 \le r \le R$, *be nonempty, closed and convex subsets of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$, *respectively. Given linear operators* $T_r : \mathbb{R}^n \to \mathbb{R}^m$, $1 \le r \le R$ *and another nonempty, closed and convex* $\Gamma \subseteq \mathbb{R}^n$, *the* ***Constrained***