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AN hp-FEM FOR SINGULARLY PERTURBED TRANSMISSION PROBLEMS*

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Abstract

We perform the analysis of the hp finite element approximation for the solution to singularly perturbed transmission problems, using *Spectral Boundary Layer Meshes.* In [12] it was shown that this method yields robust exponential convergence, as the degree p of the approximating polynomials is increased, when the error is measured in the energy norm associated with the boundary value problem. In the present article we sharpen the result by showing that the hp-Finite Element Method (FEM) on *Spectral Boundary Layer Meshes* leads to robust exponential convergence in a stronger, more *balanced* norm. Several numerical results illustrating and extending the theory are also presented.

Mathematics subject classification: 65N30.
Key words: Singularly perturbed transmission problem, Boundary layers, Interface layers, hp-FEM, Balanced norm, Exponential convergence.

1. Introduction

Singularly perturbed problems, and their numerical solution, have received much attention in recent decades (see the books [11, 17] and the references therein). One of the main difficulties in these problems is the presence of *boundary* and/or *interior layers* in the solution, whose accurate approximation, independently of the singular perturbation parameter, is vital for the overall quality of the approximate solution. For Finite Difference Methods, the robust approximation of such layers requires the use of layer adapted, parameter-dependent meshes (see e.g., [2, 20]). Such meshes have also been used in conjunction with the h version of the Finite Element Method (FEM), while for high order p/hp versions of the FEM, *Spectral Boundary Layer Meshes* from [19] (see also [5,6] and Definition 2.1 ahead) are preferred.

Singularly perturbed transmission problems are a sub-category of the aforementioned problems and their numerical approximation requires similar approaches. Their solution will exhibit the (usual) layer behavior at the boundary of the domain but it will also feature interface layers along the interface (see, e.g., [4]). In [12] we showed that the hp version of the FEM yields robust exponential convergence for singularly perturbed transmission problems, when the error

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is measured in the energy norm. However, the energy norm for some problems is rather weak in that it does not "see" the boundary layers. By this we mean that the energy norm of the layer goes to 0 as the singular perturbation parameter tends to 0, while the energy norm of the other components (of the solution) does not. This has motivated recent works by [3, 15, 16] to study the convergence of a FEM in stronger norms than the energy norm. In [10] it was shown that for reaction–diffusion problems in one– and two–dimensions, similar results can be obtained for the *hp* version FEM. In the present article we extend the analysis in a balanced norm for the *hp* version of the FEM on *Spectral Boundary Layer Meshes*, to singularly perturbed transmission problems, hence sharpening the previous results of [12]. As in [10], this analysis is inspired by corresponding work for the low order case on Shishkin meshes given in [15, 16]. We show that in the case of singularly perturbed transmission problems, estimates in a balanced norm can also be obtained for the *hp* FEM on *Spectral Boundary Layer Meshes*. Moreover, we are able to deduce, as a corollary, the maximum norm convergence of the method for such problems.

The remainder of the paper is organized as follows: in Section 2 we give the model problem and its discretization. In Section 3 we present the analysis of the *hp* FEM using the balanced norm and in Section 4 we show the results of several numerical experiments which verify (and extend) our theoretical results. Throughout the paper we will use the standard Sobolev space $H^k(\Omega)$ equipped with the norm $\|\cdot\|_{k,\Omega}$ and seminorm $|\cdot|_{k,\Omega}$. We will also use the space $H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$, where $\partial\Omega$ denotes the boundary of Ω. The $H^0(\Omega) = L^2(\Omega)$ inner-product will be denoted by $\langle\cdot,\cdot\rangle_\Omega$ and the norm of the space $L^\infty(\Omega)$ of essentially bounded functions will be denoted by $\|\cdot\|_{\infty,\Omega}$. The letters C, c, with or without subscripts, will be used to denote generic positive constants, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence. Finally, the notation $A \lesssim B$ means the existence of a positive constant C, which is independent of the quantities A and B under consideration and of any discretization or singular perturbation parameters, such that $A \leq CB$.

2. The Model Problem and its Discretization

Let $\varepsilon \in (0,1]$ be a given parameter and f a given analytic function defined in $I = (-1,1)$ and satisfying

$$\left\|f^{(n)}\right\|_{\infty,I} \leq C_f \gamma_f^n n! \quad \forall\, n \in \mathbb{N}_0, \tag{2.1}$$

for some positive constants C_f, γ_f. We consider the following transmission problem in I: Find u_ε such that

$$\begin{cases} -\varepsilon^2 \left(u_\varepsilon^-\right)'' + u_\varepsilon^- = f, & \text{in} \quad (-1,0), \\ -\left(u_\varepsilon^+\right)'' + u_\varepsilon^+ = 0, & \text{in} \quad (0,1), \\ u_\varepsilon^-(-1) = u_\varepsilon^+(1) = 0, & \\ u_\varepsilon^-(0) - u_\varepsilon^+(0) = 0, & \\ \varepsilon^2 \left(u_\varepsilon^-\right)'(0) - \left(u_\varepsilon^+\right)'(0) = 0, & \end{cases} \tag{2.2}$$

where u_ε^- (resp. u_ε^+) means the restriction of u_ε to $(-1,0)$ (resp. $(0,1)$). Note that in this problem the small parameter ε appears only on $(-1,0)$. Consequently the formal limit problem