Journal of Computational Mathematics
Vol.35, No.3, 2017, 245–264.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1605-m2015-0397

DISCRETE MAXIMUM PRINCIPLE AND CONVERGENCE OF POISSON PROBLEM FOR THE FINITE POINT METHOD*

Guixia Lv
Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P. O. Box 8009-26, Beijing 100088, China
Email: lv_guixia@iapcm.ac.cn
Shunkai Sun
Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing 100088, China
Email: sun_shunkai@iapcm.ac.cn
Longjun Shen
Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P. O. Box 8009-26, Beijing 100088, China
Email: shenlj@iapcm.ac.cn

Abstract

This paper makes some mathematical analyses for the finite point method based on directional difference. By virtue of the explicit expressions of numerical formulae using only five neighboring points for computing first-order and second-order directional differentials, a new methodology is presented to discretize the Laplacian operator defined on 2D scattered point distributions. Some sufficient conditions with very weak limitations are obtained, under which the resulted schemes are positive schemes. As a consequence, the discrete maximum principle is proved, and the first order convergent result of $O(h)$ is achieved for the nodal solutions defined on scattered point distributions, which can be raised up to $O(h^2)$ on uniform point distributions.

Mathematics subject classification: 65D25, 65N06, 65N12, 65N15.
Key words: Finite point method, Directional difference, Meshless, Discrete maximum principle, Convergence analysis.

1. Introduction

In the last few decades, motivated by practical applications, meshless methods, as an alternative class of methods to classical mesh methods, have emerged and made great progress in scientific and engineering computational fields (see, e.g., [1-5], and references therein). However, as compared with application aspects of meshless methods, rigorous mathematical theories are still rather scarcely.

Without mesh restriction, meshless methods have distinguished advantages in flexibility and adaptivity, which on the other hand result in many difficulties in meshless mathematical analysis. As pointed out in paper [6], in mathematical analyses of meshless methods in weak form [7-10], the main difficulty arises in constructing a test function space for the boundary problems with Dirichlet conditions. Once this difficulty is overcome, the remaining part of the convergence analysis follows the idea similar to that of the finite element method. In contrast to meshless methods in weak form, as there is no directly available theory of function spaces,

* Received September 18, 2015 / Revised version received March 25, 2016 / Accepted May 18, 2016 / Published online April 25, 2017 /

it is more complicated and challenging to supply the mathematical analysis to the meshless methods in strong form. In this aspect, only a few papers present the mathematical convergence for numerical solutions [6, 11]. [6] defines a point collocation scheme for the Poisson problem, and proves the discrete maximum principle only for some special point distributions. [11] gives error estimates for the finite point method under certain conditions imposed on the nodes and shape functions. Thus, the present paper is to make mathematical analysis to a strong-form meshless method on scattered point distributions.

We consider the finite point method (FPM) proposed by Shen [12] (see also [13]). It is based on the directional difference, and falls into the strong-form meshless category. It distinguishes from the FPM first proposed by Oñate [14] in that it employs just adequate points to solve for derivatives in terms of given order, while the FPM [14] employs much more points than unknowns resulting an overdetermined system solved by the weighted least-square procedure. In [12], explicit numerical formulae for approximations to directional differentials are derived with expected accuracy by using the information of proper scattered points, which is favorable to making the theoretical analyses. By this, solvability conditions of numerical derivatives are rigorously given, which give general guiding principles for selecting neighboring points with definite geometry configuration.

In the present paper, by virtue of the explicit expressions of the numerical formulae of directional derivatives [12], we present a methodology to discretize the Laplacian operator defined on 2D scattered point distributions, and give some sufficient conditions with very weak limitations for positive schemes of Laplacian operator. To prove the resulted scheme satisfying the discrete maximum principle, we state a new definition of the connectivity of discrete point sets differing from the classical definition of that for mesh methods. In fact, for mesh methods, the discrete system often results from uniform discrete schemes defined on different meshes, while for meshless methods situations may be more complex, because we can have different types of discrete schemes at different discrete points. This discrepancy arises mainly by their different neighborhood criteria. The fixed neighborhood criteria is often adopted in mesh methods due to limitation of the mesh structure, while the more flexible neighborhood criteria is explored in meshless methods, which is one of the notable characteristics of meshless methods, however, it also results in more difficulties in mathematical analysis as compared with mesh methods.

Based on the proof of the discrete maximum principle, we obtain the convergent result of the discrete problem. The nodal solutions defined on scattered point distributions are proved to have the first order convergence result of $O(h)$, which can be raised up to $O(h^2)$ on uniform point distributions.

The paper is organized as follows. Section 2 presents some preliminaries including numerical formulae of directional derivatives. Section 3 develops a methodology to discretize the Laplacian operator. Section 4 gives sufficient conditions for positive schemes. Section 5 proposes the concept of connected point set, and proves the discrete maximum principle and the convergent result for the discrete problem. Finally, some concluding remarks are made in Section 6.

2. Preliminaries

2.1. Denotations and definitions

To simplify presentation, we first introduce some denotations and definitions as defined in [12]. Let us denote by