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# LINEARLY CONVERGENT FIRST-ORDER ALGORITHMS FOR SEMIDEFINITE PROGRAMMING*

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**Abstract**

In this paper, we consider two different formulations (one is smooth and the other one is nonsmooth) for solving linear matrix inequalities (LMIs), an important class of semidefinite programming (SDP), under a certain Slater constraint qualification assumption. We then propose two first-order methods, one based on subgradient method and the other based on Nesterov's optimal method, and show that they converge linearly for solving these formulations. Moreover, we introduce an accelerated prox-level method which converges linearly uniformly for both smooth and non-smooth problems without requiring the input of any problem parameters. Finally, we consider a special case of LMIs, i.e., linear system of inequalities, and show that a linearly convergent algorithm can be obtained under a much weaker assumption.

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*Key words:* Semi-definite Programming, Linear Matrix Inequalities, Error Bounds, Linear Convergence.

## 1. Introduction

Semidefinite Programming (SDP) is one of most interesting branches of mathematical programming in last twenty years. Semidefinite Programming can be used to model many practical problems in various fields such as convex constrained optimization, combinatorial optimization, and control theory, please refer to [1] for a general survey on SDP. Algorithms for solving S-DP have been explosively studied since the landmarking works were made by Nesterov and Nemirovski [2–5], in which they showed that the interior point (IP) methods for linear programming (LP) can be extended to SDP (related topics can be found in [6–10]). Despite the fact that SDP can be solved in polynomial time by IP methods, the cost per iteration for these methods become prohibitively large as the number of variables or constraints increase. The recent research during the last few years has been focused on first-order methods due to their reduced iteration cost for solving large scale SDP (e.g., Nesterov's optimal methods [11, 12], Nemirovski's mirror-prox method [13], and spectral bundle methods [14]).

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In [15], Lan, Lu and Monteiro proposed a class of first-order methods which exhibit an $\mathcal{O}(1/\epsilon)$ iteration complexity for solving large-scale semidefinite programming. The basic idea is to reformulate the primal and dual SDP problems as a linear matrix inequality (LMI) and then apply Nesterov's method [11,12] or Nemirovski's mirror-prox method [13] to solve the resulting reformulation. Their work shows the possibility of solving general SDPs by developing efficient algorithms for solving LMIs. It is also worth noting that LMIs are important modeling tools in their own right and have been widely studied in system and control theory, system identification and signal processing [16,17]. In [18,19], the positive definite variable $X$ is replaced by $RR^\top$ in the primal augmented Lagrangian framework, then the limited-memory BFGS method is applied to minimize each augmented Lagrangian function. In [20,21], the dual augmented Lagrangian function is first reformulated by using a projection operator, then it is minimized by a semismooth Newton approach combined with the conjugate gradient method. An alternating direction method of multipliers [22] also minimizes the dual augmented Lagrangian function sequentially with respect to the Lagrange multipliers corresponding to the linear constraints, then the dual slack variables and finally the primal variables, while in each minimization keeping the other variables fixed.

Inspired by the importance of SDP and the related LMIs, we propose in this paper a new class of first-order methods that converge linearly to solve LMIs. Our development utilizes the error bounding techniques that have been intensively studied in nonlinear optimization (e.g., Luo and Tseng [23–25] on the error bounds for solving a variety of nonlinear optimization problems, Zhang [26] on the error bounds for general convex conic problem under some various conditions, Deng and Hu [27] and Jourani and Ye [28] on the error bound for semidefinite programming, and other related topics in [29, 6, 7]). More specifically, consider the optimization problem of

$$f^* = \min_{x \in X} f(x), \tag{1.1}$$

where $X \subseteq \mathbb{R}^n$ is a closed convex set and $f : X \to \mathbb{R}$ is a closed convex function. Let $X^*$ be the set of optimal solutions and denote

$$d(x, X^*) := \min_{y \in X^*} \|x - y\|$$

These error bounds provide, under certain slater conditions, the relationship between the distance to the set of optimal solutions and the function values, e.g.,

$$d(x, X^*) \leq \mu[f(x) - f^*]$$

for some $\mu > 0$. It should be noted, however, that the value of $\mu$ is usually unknown and difficult to estimate.

Our contribution in this paper mainly lies on the following several aspects. Firstly, we propose a nonsmooth reformulation for LMIs, given in the form of (1.1) with objective function being a nonsmooth convex function. By properly restarting the classical subgradient method, we show that the iteration complexity for solving SDP can be bounded by $\mathcal{O}(\mu^2 \log(1/\epsilon))$. Each iteration of this method involves only a maximum eigenvalue decomposition. Secondly, in order to improve the previous iteration complexity bound, we study a smooth reformulation of LMIs, still given in the form of (1.1), but with a smooth objective function $f$. We show that by properly restarting Nesterov's optimal method, the iteration complexity for solving the smooth reformulation of LMIs can be bounded by $\mathcal{O}(\mu \log(1/\epsilon))$, but each iteration of this algorithm requires a full eigenvalue decomposition. Thirdly, while the aforementioned methods require