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A MULTIGRID SEMISMOOTH NEWTON METHOD FOR SEMILINEAR CONTACT PROBLEMS *

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Abstract

This paper develops and analyzes multigrid semismooth Newton methods for a class of inequality-constrained optimization problems in function space which are motivated by and include linear elastic contact problems of Signorini type. We show that after a suitable Moreau-Yosida type regularization of the problem superlinear local convergence is obtained for a class of semismooth Newton methods. In addition, estimates for the order of the error introduced by the regularization are derived. The main part of the paper is devoted to the analysis of a multilevel preconditioner for the semismooth Newton system. We prove a rigorous bound for the contraction rate of the multigrid cycle which is robust with respect to sufficiently small regularization parameters and the number of grid levels. Moreover, it applies to adaptively refined grids. The paper concludes with numerical results.

Mathematics subject classification: 65K10, 65C20, 65N30, 65N55, 74M15.
Key words: Contact problems, Semismooth Newton methods, Multigrid methods, Error estimates.

1. Introduction

In this paper, a class of multigrid semismooth Newton methods for constrained optimization problems is developed and systematically analyzed. The considered problem class is motivated by linear elastic contact problems, which are included as a special case. We work with the same nonpenetration constraints as in linear contact, but compared to linear elasticity, we cover more general cost functions than the quadratic elastic energy. More precisely, the problems have the following form:

$$\min_{u\in\mathbf{U}} J(u) \quad \text{subject to} \quad \tau_C^n(u) \le \psi \quad \text{on } \Gamma_C. \tag{1.1}$$

Here, $\mathbf{U} = \{u \in H^1(\Omega)^d : \tau_D u = 0\}$, $\Omega \subset \mathbb{R}^d$ is a bounded open domain, and Γ_C and Γ_D are disjoint subsets of the boundary $\partial\Omega$ of Ω. Further, $\tau_D : H^1(\Omega)^d \to H^{1/2}(\Gamma_D)^d$ is the trace operator on Γ_D, i.e., $\tau_D(u)(x) = u(x)$ for all $x \in \Gamma_D$ if u is continuous on $\bar{\Omega}$; $\tau_C^n = n^T\tau_C : \mathbf{U} \to V := H^{1/2}(\Gamma_C)$ is the normal trace operator on Γ_C, with n denoting the outer unit normal; further, $\psi : \Gamma_C \to \mathbb{R}$ is a given, sufficiently smooth function. Targeting for Newton-type

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methods, the objective function $J : \mathbf{U} \to \mathbb{R}$ is assumed to be twice continuously differentiable. Some additional requirements on Ω, Γ_C, Γ_D, and J will be given below.

Problem of the form (1.1) arise, e.g., in elastic contact problems, where Ω is the reference configuration of an elastic body, $u(x)$ denotes the displacement of the reference point $x \in \Omega$, and J is the total energy. The constraint then expresses that the normal displacement on Γ_C shall not exceed ψ, which can be interpreted as the normal distance to a rigid obstacle.

Our approach uses a Moreau-Yosida (MY) regularization to obtain a nonsmooth approximation of the first order optimality conditions that is suitable for applying a semismooth Newton method. We develop a superlinear convergence theory in function space as well as error estimates for the MY-regularized solutions. A particular focus of the paper is put on a multigrid method for preconditioning or solving the linear semismooth Newton systems.

The MY-regularization is required since the problem contains a pointwise inequality constraint that is posed in a Sobolev space $V := H^{1/2}(\Gamma_C)$. The natural space for the Lagrange multiplier is then the dual space V' and thus the complementarity condition cannot be written in a pointwise almost everywhere form. For sufficiently smooth data, regularity results for the solution can be used to infer that the multiplier is an L^q-function. Then, a nonsmooth pointwise reformulation of the complementarity condition would in fact be possible. However, in a primal-dual formulation of the optimality system, replacing the multiplier space V' by L^q does not provide a framework where the linear operator in the Newton system is boundedly invertible. Thus, a dual regularization would be required to fix this [1,2]; as we will see, such a regularization is equivalent to the Moreau-Yosida approach, see also [3, Sec. 2.1] and [2, Sec. 8.2.4 and 9.2]. A different alternative, chosen, e.g., in [4–7], is to consider the problem after discretization and relying on the fact that then all norms are equivalent. However, this comes at the cost of dimension-dependent condition numbers and norm equivalence constants. This regularization by discretization (or well-posedness through discretization) strategy requires to combine it with a nested iteration from coarse to fine grids to compensate for the lack of mesh-independence since a function space counterpart of the discrete algorithm is then missing. We therefore prefer to work with the MY-regularization, which by our error estimates can be balanced with the discretization error, to have a well-posed algorithm also in function space.

Extending results in [1–3], we show that a regularization with parameter $\alpha > 0$ results in a solution that deviates at most by $o(\alpha^{1/2})$ (as $\alpha \to 0^+$) from the true solution if the true Lagrange multiplier is in $L^2(\Gamma_C)$. Further, if the Lagrange multiplier is in $H^s(\Gamma_C)$, $0 < s \leq 1/2$ and the derivative of J is κ-Hölder continuous near the solution (which holds globally with $\kappa = 1$ for linear elasticity), we show that the convergence rate is $O(\alpha^{\frac{1+2s}{2+4s(1-\kappa)}})$. Multiplier regularity can be ensured under suitable assumptions by invoking regularity results for elastic obstacle problems [8–10]. We then introduce a finite element discretization and, based on this, a discrete counterpart of the semismooth Newton's method.

The main part of the paper is devoted to the analysis of a multigrid cycle that can be used stand alone or as a preconditioner to solve the semismooth Newton system to the desired accuracy. Due to the regularization, multigrid methods for the semismooth Newton system require special care. The regularization introduces an algebraic (i.e. non-differential) operator acting on Γ_C that is strongly weighted. This requires to develop a special multigrid iteration. Building on a general framework of multilevel convergence theory [11], we prove a guaranteed contraction rate that is independent of the number of grid levels and uniform for all regularization parameters $\alpha \in (0, \alpha_h^+]$, where the upper bound $\alpha_h^+ > 0$ depends on the mesh size h of the finest grid in the contact region, but is larger than required to balance the regularization