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A LINEARLY-FITTED CONSERVATIVE (DISSIPATIVE) SCHEME FOR EFFICIENTLY SOLVING CONSERVATIVE (DISSIPATIVE) NONLINEAR WAVE PDES*

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Abstract

The extended discrete gradient method is an extension of traditional discrete gradient method, which is specially designed to solve oscillatory Hamiltonian systems efficiently while preserving their energy exactly. In this paper, based on the extended discrete gradient method, we present an efficient approach to devising novel schemes for numerically solving conservative (dissipative) nonlinear wave partial differential equations. The new scheme can preserve the energy exactly for conservative wave equations. With a minor remedy to the extended discrete gradient method, the new scheme is applicable to dissipative wave equations. Moreover, it can preserve the dissipation structure for the dissipative wave equation as well. Another important property of the new scheme is that it is linearly-fitted, which guarantees much fast convergence for the fixed-point iteration which is required by an energy-preserving integrator. The efficiency of the new scheme is demonstrated by some numerical examples.

Mathematics subject classification: 65L05, 65L07, 65L20, 65P10, 34C15.
Key words: Conservative (dissipative) wave PDEs; Structure-preserving algorithm; Linearly-fitted; Average Vector Field formula; Sine-Gordon equation.

1. Introduction

Numerical schemes that conserve geometric structure have been shown to be useful when studying the long-time behaviour of dynamical systems. Such schemes are sometimes called geometric or structure-preserving integrators. The structure includes physical/geometric properties such as first integrals, symplecticity, symmetries and reversing symmetries, phase-space volume, Lyapunov functions, foliations. Geometric algorithms have important applications in many fields, such as fluid dynamics, celestial mechanics, molecular dynamics, quantum physics, plasma physics, quantum mechanics, and meteorology. We refer the reader to [1–3] for recent

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surveys of this research. It has now become a common practice that the consideration of qualitative properties in ordinary and partial differential equations is important when designing numerical schemes. For ordinary differential equations (ODEs) it is possible to devise relatively general frameworks for structure preservation. This seems somewhat much more difficult for partial differential equations (PDEs) because PDEs are a huge and motley collection of problems and each equation under consideration normally requires a dedicated scheme (see, e.g. [6–10]). Fortunately, many attempts have been made to give a fairly general methodology to develop geometric schemes for PDEs. For example, in [4], by discretizing the energy of the PDEs to get an ODE system, then applying the average vector field method to the resulting system, the authors proposed a systematic procedure to deal with evolutionary PDEs as far as conservation or dissipation of energy is concerned. Another example is the PDEs that can be formulated into multi-symplectic form to which, one can apply a scheme which preserves a discrete version of this form (see, e.g. [5], for a review of this approach). Many enery-preserving or multi-symplectic methods are derived for Hamiltonian PDEs based on the multi-symplectic formulation (see, e.g., [11–14]).

In recent years, there has been an enormous advance in dealing with the oscillatory systems

$$\ddot{q} + Mq = f(q), \tag{1.1}$$

which can be obtained by spatial semi-discretization of wave equations and some useful approaches to constructing Runge-Kutta-Nyström (RKN)-type integrators have been proposed (see, e.g. [15–20]). Very recently, taking account of the special structure introduced by the linear term Mq, Wu *et al.* [20] formulated a standard form of the multidimensional extended RKN (ERKN) integrators. The ERKN integrators exhibit the correct qualitative behaviour much better than classical RKN methods due to using the special structure of the equation brought by the linear term Mq. For further work on this topic, we refer the reader to [19, 21, 22]. If f is the negative gradient of a scalar function V, i.e., $f = -\nabla V$, then (1.1) is a multi-frequency oscillatory Hamiltonian system. In [23], integrating the idea of the discrete gradient method with the ERKN integrator, the authors presented an extended discrete gradient formula for the oscillatory Hamiltonian system (1.1).

In this paper, we will propose and investigate an efficient approach to dealing with nonlinear wave PDEs following the line of [4]. Firstly, by approximating the functional whose negative variational derivative is the right-hand side term of the underlying wave equation, we semi-discretize the conservative wave equations into a Hamiltonian system of ODEs or the dissipative wave equations into a dissipative system of ODEs. We then apply the extended discrete gradient method to the resulting system of ODEs. This process gives a conservative scheme for conservative wave PDEs and a dissipative scheme for dissipative wave PDEs, and can be applied to a large scope of wave equations in a systematic way.

The outline of this paper is as follows. The preliminaries are given in Section 2. In Section 3, we recall the extended discrete gradient method, based on which a new dissipative scheme is proposed for dissipative systems with a damping term. In Section 4, the new numerical schemes are applied to some conservative/dissipative wave equations to show the efficiency and robustness in comparison with the existing methods in the literature. The last section focuses on some conclusions and discussions.