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TIME DOMAIN BOUNDARY ELEMENT METHODS FOR THE NEUMANN PROBLEM: ERROR ESTIMATES AND ACOUSTIC PROBLEMS*

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Abstract

We investigate time domain boundary element methods for the wave equation in $\mathbb{R}^3$, with a view towards sound emission problems in computational acoustics. The Neumann problem is reduced to a time dependent integral equation for the hypersingular operator, and we present a priori and a posteriori error estimates for conforming Galerkin approximations in the more general case of a screen. Numerical experiments validate the convergence of our boundary element scheme and compare it with the numerical approximations obtained from an integral equation of the second kind. Computations in a half-space illustrate the influence of the reflection properties of a flat street.

Mathematics subject classification: 65N38, 65R20, 74J05.
Key words: Time domain boundary element method, Wave equation, Neumann problem, Error estimates, Sound radiation.

1. Introduction

Motivated by the sound radiation of tires [2], this article analyzes time domain boundary element methods for a scattering or emission problem for the wave equation outside a sound-hard obstacle.

Let $d \geq 2$ and $\Omega^i \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We aim to find a weak solution to an acoustic initial boundary problem for the wave equation in $\Omega^e = \mathbb{R}^d \setminus \overline{\Omega^i}$:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \qquad \text{in } \mathbb{R}^+ \times \Omega^e, \tag{1.1a}$$

$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0 \qquad \text{in } \Omega^e, \tag{1.1b}$$

$$\frac{\partial u}{\partial n} = \tilde{g} \qquad \text{on } \mathbb{R}^+ \times \Gamma\ . \tag{1.1c}$$

Here n denotes the inward unit normal vector to $\Gamma = \partial\Omega^e$, and $2\tilde{g} = g$ lies in a suitable Sobolev space.

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This article reduces the boundary problem (1.1) to a time dependent integral equation on $\mathbb{R}^+ \times \Gamma$ and studies Galerkin time domain boundary element methods for its approximation. While we focus on the hypersingular integral equation, numerical examples compare it to an integral equation of the second kind.

Time domain boundary integral formulations for hyperbolic equations and their numerical solution were introduced by Friedman and Shaw [7], resp. Cruse and Rizzo [4]. A first mathematical analysis of time dependent boundary element methods goes back to Bamberger and Ha-Duong [1,12], see also [9] for Dirichlet and acoustic boundary problems in a half-space. First numerical experiments for integral equations of the second kind in the full space were reported by Ding et al. [5], and the practical realization of the numerical marching-on-in-time scheme include the Ph.D. thesis of Terrasse [19] as well as [14]. Also, fast collocation methods have been developed in the engineering literature [21]. Some recent work around space-time adaptive methods and applications is surveyed in [8]. A detailed exposition of the mathematical background of time domain integral equations and their discretizations is available in the lecture notes by Sayas [18].

In this work we investigate the Neumann problem (1.1), present a priori and a posteriori error estimates for the Galerkin solution of the time dependent hypersingular integral equation of the first kind (with the normal derivative of the double layer potential). We compare the numerical scheme for the hypersingular equation with numerical approximations of an integral equation of the second kind (with the normal derivative of the single layer potential). We analyze the integral equations in the more general setting of a screen Γ, i.e., allow $\partial\Gamma \neq \emptyset$, which will prove relevant for work in progress on dynamic contact problems.

A motivation for these results comes from applications to traffic noise [2,9,10], where adaptive methods based on a posteriori error estimates are crucial to resolve singular geometries. With this application in mind, we also present numerical results in an acoustic half-space. Here, $\Omega^i \subset \mathbb{R}^d_+$ is a bounded domain with $\mathbb{R}^d_+ \setminus \overline{\Omega^i}$ Lipschitz, and the Neumann boundary conditions on $\Gamma = \partial\Omega^i \cap \mathbb{R}^d_+$ are supplemented by acoustic boundary conditions

$$\frac{\partial u}{\partial n} - \alpha \frac{\partial u}{\partial t} = 0 \tag{1.2}$$

on $\mathbb{R}^{d-1} \times \{0\} = \partial\mathbb{R}^d_+$, $\alpha \geq 0$. Screens arise naturally when $\partial\Omega^i \cap \partial\mathbb{R}^d_+ \neq \emptyset$.

Notation: To simplify notation, we will write $f \lesssim g$, if there exists a constant $C > 0$ independent of the arguments of the functions f and g such that $f \leq Cg$. We will write $f \lesssim_\sigma g$, if C may depend on σ.

2. Time-domain Integral Equations and Discretization

2.1. Boundary integral equations

Space–time anisotropic Sobolev spaces on the boundary Γ provide a convenient setting to study the mapping properties of the time-dependent layer operators [3,13]. We more generally consider the case of a screen, where the orientable, $(d-1)$-dimensional Lipschitz submanifold $\Gamma \subset \mathbb{R}^d$ may have a boundary. If $\partial\Gamma \neq \emptyset$, first extend Γ to a closed, orientable manifold $\widetilde{\Gamma}$.

For $\sigma > 0$, $s, r \in \mathbb{R}$ the space $H^s_\sigma(\mathbb{R}^+, H^r(\widetilde{\Gamma}))$ consists of certain distributions ϕ on $\mathbb{R}^+ \times \widetilde{\Gamma}$, vanishing at $t = 0$, such that in local coordinates the space–time Fourier–Laplace transform $\mathcal{F}\phi$