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ON ADAPTIVE WAVELET BOUNDARY ELEMENT METHODS *

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Abstract

The present article is concerned with the numerical solution of boundary integral equations by an adaptive wavelet boundary element method. This method approximates the solution with a computational complexity that is proportional to the solution's best N-term approximation. The focus of this article is on algorithmic issues which includes the crucial building blocks and details about the efficient implementation. By numerical examples for the Laplace equation and the Helmholtz equation, solved for different geometries and right-hand sides, we validate the feasibility and efficiency of the adaptive wavelet boundary element method.

Mathematics subject classification: 41A25, 65N38, 65T60.
Key words: Boundary element method, wavelets, adaptivity.

1. Introduction

In science and engineering, one often comes across partial differential equations, some of which can be formulated as boundary integral equations on the boundary of the domain of interest. Solving the original problem would result in having to discretize the problem in a domain (e.g. with finite element methods), which would lead to a sparse but extremely large system of linear equations, especially in the three-dimensional situation. Rewriting the problem by a boundary integral equation not only reduces the dimensionality by one, but does give the possibility to solve also exterior boundary value problems. Particularly for such problems, this approach brings many advantages, since it is not necessary to find a way (e.g. by introducing artificial boundaries) to handle the infinite expansion of the domain. Of course, this advantage does not come entirely without cost. Since the involved operators are not local, the resulting matrices are dense and the complexity to solve the linear system by the boundary element method is at least $\mathcal{O}(N^2)$, with N denoting the degrees of freedom.

Modern approaches like the *fast multipole method* [1,2], the *panel clustering* [3], the *adaptive cross approximation* [4, 5], or *hierarchical matrices* [6, 7] are known to reduce the complexity to log-linear or even linear cost. Another approach is wavelet matrix compression [8]. The wavelets' vanishing moments lead, in combination with the fact that the kernel of the integral operator becomes smoother when getting farther away from the diagonal, to a quasi-sparse system matrix. As shown in [9], only $\mathcal{O}(N)$ matrix entries are relevant for maintaining the convergence rate of the underlying Galerkin scheme.

A further issue to be addressed for the efficient solution of boundary integral equations is the one of adaptivity. For non-smooth geometries or right-hand sides, it is necessary being able to resolve specific parts of the geometry, while other parts could stay coarse. In contrast to

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uniform refinement, an adaptive refinement reduces the degrees of freedom drastically without compromising the accuracy. This means that not only a lot of computation power can be saved, but also a lot of memory, making the computation of certain problems possible in the first place. Efficient and reliable a posteriori error estimators have first been introduced in [10] and convergence of adaptive refinements for traditional boundary element methods has been established in [11, 12]. But we are not aware of an implementation which combines these error estimators with fast boundary element methods.

We thus follow here an different approach which has been proposed in [13, 14] for local operators and in [9, 15] for nonlocal operators. Namely, we cast the boundary integral equation into an infinite system of linear equations and solve it then approximately by an iterative method. As the application of the infinite system matrix has to be approximated during the solution process, we have to choose a certain portion out of this infinite matrix. Hence, refinement rather means that more wavelets are added. In fact, we aim at choosing the N wavelets which will contribute most to the approximate solution. This concept is referred to as the *best N-term approximation*, see e.g. [16].

Where the efficient computation of the matrix entries is the most demanding and time consuming part of the whole implementation, it is not possible to achieve efficiency without using the appropriate adaptive structures. It is absolutely necessary to work with element and wavelet trees as already proposed in [17, 18]. We will introduce related building blocks Rhs, Coarse, Apply, and Solve, which are already known in theory from e.g. [13, 14, 18, 19]. The implementation of these routines is discussed in the present context of boundary element methods. The numerical method is able to compute the solution of the boundary integral equation in asymptotically optimal complexity. This means that any target accuracy can be achieved at a computational expense that stays proportional to the number of degrees of freedom (within the setting determined by an underlying wavelet basis) that would ideally be necessary for realizing that target accuracy if full knowledge about the unknown solution were given.

In this article, besides presenting results for the single-layer operator of the Laplacian, which is symmetric and positive definite, we also present results for the Brackhage-Werner formulation of the (low-frequency) Helmholtz equation. We thus arrive at a linear combination of the acoustic single-layer operator and the acoustic double-layer operator. Here, the theory of [9, 15] does not hold anymore. Nevertheless, the arguments of [20] are applicable for proving optimality of the underlying adaptive scheme since the operator under consideration is a compact perturbation of a symmetric and positive definite operator.

The outline of this article is as follows. At first, in Section 2, we introduce the boundary integral equation and the surface representation under consideration. Then, in Section 3, we present the piecewise constant wavelet basis which we will employ to cast the boundary integral equation into an equivalent, bi-infinite system of linear equations. Section 4 is dedicated to the realization of an adaptive algorithm of optimal complexity. Numerical results are given in Section 5. Finally, in Section 6, we state concluding remarks.

2. Boundary Integral Equations and Surface Representation

Let $\Omega \subset \mathbb{R}^3$ be a bounded and simply connected domain. Its boundary $\Gamma := \partial\Omega$ is assumed to be composed by a union of smooth, four-sided patches Γ_i:

$$\Gamma = \bigcup_{i=1}^{M} \Gamma_i, \quad \Gamma_i = \boldsymbol{\gamma}_i(\square) \quad i = 1, \ldots, M.$$