Journal of Computational Mathematics
Vol.36, No.2, 2018, 159–182.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1607-m2016-0539

WEAK ERROR ESTIMATES FOR TRAJECTORIES OF SPDEs UNDER SPECTRAL GALERKIN DISCRETIZATION*

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Abstract

We consider stochastic semi-linear evolution equations which are driven by additive, spatially correlated, Wiener noise, and in particular consider problems of heat equation (analytic semigroup) and damped-driven wave equations (bounded semigroup) type. We discretize these equations by means of a spectral Galerkin projection, and we study the approximation of the probability distribution of the trajectories: test functions are regular, but depend on the values of the process on the interval $[0,T]$.

We introduce a new approach in the context of quantative weak error analysis for discretization of SPDEs. The weak error is formulated using a deterministic function (Itô map) of the stochastic convolution found when the nonlinear term is dropped. The regularity properties of the Itô map are exploited, and in particular second-order Taylor expansions employed, to transfer the error from spectral approximation of the stochastic convolution into the weak error of interest.

We prove that the weak rate of convergence is twice the strong rate of convergence in two situations. First, we assume that the covariance operator commutes with the generator of the semigroup: the first order term in the weak error expansion cancels out thanks to an independence property. Second, we remove the commuting assumption, and extend the previous result, thanks to the analysis of a new error term depending on a commutator.

Mathematics subject classification: 60H15, 65C30.
Key words: Stochastic Partial Differential Equations, Weak approximation, Spectral Galerkin discretization

1. Introduction

The numerical analysis of stochastic differential equations (SDEs), in both the weak and strong senses, has been an active area of research over the last three decades [15, 21]. The analysis of numerical methods for stochastic partial differential equations (SPDEs) has attracted a lot of attention and in recent years a number of texts have appeared in this field; see for instance the recent monographs [13, 18, 19]. The aim of this article is to give a simple argument

* Received February 13, 2016 / Revised version received June 6, 2016 / Accepted July 8, 2016 / Published online February 5, 2018 /

allowing to relate the weak order to the strong order of convergence on the space of trajectories for a class of spatial approximations to SPDEs.

We focus on the following class of semilinear SPDEs, written using the stochastic evolution equations framework in Hilbert spaces, from [6]:

$$dX(t) = \mathcal{A}X(t)dt + \mathcal{F}(X(t))\,dt + d\mathcal{W}^{\mathcal{Q}}(t),\ X(0) = x_0. \tag{1.1}$$

The semi-linear equation (1.1) is driven by an additive Wiener process $\mathcal{W}^{\mathcal{Q}}$, where $\mathcal{Q}$ is a covariance operator. The following parabolic, resp. hyperbolic, SPDEs can be written as (1.1), with appropriate definitions of the coefficients $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{Q}$ in terms of A, F and Q:

- *the semi-linear stochastic heat equation* (parabolic case), with $X = u$,

$$du(t) = Au(t)dt + F(u(t))\,dt + dW^{Q}(t),\ u(0) = u_0; \tag{1.2}$$

- *the damped-driven semi-linear wave equation* (hyperbolic case), with $X = (u, v)$

$$\begin{cases} du(t) = v(t)dt \\ dv(t) = -\gamma v(t)dt + Au(t)dt + F(u(t))dt + dW^{Q}(t). \end{cases} \tag{1.3}$$

These two equations will be the focus of our work. Notation and assumptions on the coefficients are precised in Section 2 below. For simplicity, in this introductory section, we assume that $F : H \to H$ is of class $\mathcal{C}^2$.

The solution X of (1.1) (well-posed under assumptions given below) is a continuous-time stochastic process taking values in a separable, infinite-dimensional Hilbert space, which we denote by H. As for deterministic PDE problems, two kinds of discretizations are required in order to build practical algorithms: a time-discretization, which in the stochastic context is often a variant of the Euler-Maruyama method, and a space-discretization, which is based on finite differences, finite elements or spectral approximation. In this article, we only study the space-discretization error (no time-discretization), using a spectral Galerkin projection, *i.e.* by projecting the equation on vector spaces spanned by N eigenvectors of the linear operator A. Precisely, X is approximated by the solution X_N of an equation of the form

$$dX_N(t) = \mathcal{A}_N X_N(t)dt + \mathcal{F}_N(X_N(t))\,dt + d\mathcal{W}^{\mathcal{Q}_N}(t), \tag{1.4}$$

where the coefficients $\mathcal{A}_N$, $\mathcal{F}_N$, $\mathcal{Q}_N$ and the initial condition $X_N(0)$ are defined using the orthogonal projection $P_N \in \mathcal{L}(H)$ onto the N-dimensional vector space spanned by $e_1, \ldots, e_N$, where $Ae_n = -\lambda_n e_n$, for all $n \in \mathbb{N}$, with $\lambda_{n+1} \geq \lambda_n \geq \lambda_1 > 0$.

When looking at rates of convergence for the discretization of SPDEs, the metric one uses to compare random variables plays an important role. Let $\mathcal{Z}$, resp. $(\mathcal{Z}_n)_{n\in\{1,2,\ldots\}}$, denote a random variable, respectively a sequence of random variables, defined on a probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$, with values in a Polish space E (separable and complete metric space, with distance denoted by d_E). Strong approximation is a pathwise concept, typically defined through convergence in the mean-square sense of $\mathcal{Z}_n$ to $\mathcal{Z}$, *i.e.* the convergence of the strong error

$$e_n^{\text{strong}} = \left(\mathbb{E}d_E(\mathcal{Z}, \mathcal{Z}_n)^2\right)^{1/2},$$

or in an almost sure sense; see [15] for details. Weak approximation corresponds to convergence in distribution of $\mathcal{Z}_n$ to $\mathcal{Z}$, which is often encoded in a weak error of the type

$$e_n^{\text{weak}} = \sup_{\varphi\in\mathcal{C}} |\mathbb{E}\varphi(\mathcal{Z}) - \mathbb{E}\varphi(\mathcal{Z}_n)|,$$