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ON EFFECTIVE STOCHASTIC GALERKIN FINITE ELEMENT METHOD FOR STOCHASTIC OPTIMAL CONTROL GOVERNED BY INTEGRAL-DIFFERENTIAL EQUATIONS WITH RANDOM COEFFICIENTS*

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Abstract

In this paper, we apply stochastic Galerkin finite element methods to the optimal control problem governed by an elliptic integral-differential PDEs with random field. The control problem has the control constraints of obstacle type. A new gradient algorithm based on the pre-conditioner conjugate gradient algorithm (PCG) is developed for this optimal control problem. This algorithm can transform a part of the state equation matrix and co-state equation matrix into block diagonal matrix and then solve the optimal control systems iteratively. The proof of convergence for this algorithm is also discussed. Finally numerical examples of a medial size are presented to illustrate our theoretical results.

Mathematics subject classification: 65N06, 65B99.
Key words: Effective gradient algorithm, Stochastic Galerkin method, Optimal control problem, Elliptic integro-differential equations with random coefficients.

1. Introduction

Optimal control problems governed by partial different equations have been a major research topic in applied mathematics and control theory. The finite element approximation of optimal control plays an important role in numerical methods for these problems. There have been extensive studies on this topic for such as elliptic equations, parabolic equations, Stokes equations, Navier-Stokes equation and Integro-differential equations. Some of recent progressesin this area have been summarized in [21, 26, 30, 34, 43, 47, 52, 53, 55, 58], and the references cited therein, and the PPGA algorithm given in [34] has been verified as an effective algorithm for the constraint optimal control problems.

Uncertainty, such as uncertain parameters, arises in many complex real-world problems of physical and engineering interests. It is well known that these problems can be described by different kinds of stochastic partial differential equations (SPDEs). Among them, integro-differential equations and their control of this nature appear in applications such as heat conduction in materials with memory, population dynamics, and viscous-elasticity; cf., e.g., Friedman

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and Shinbrot [12], Heard [22], and Renardy et al. [49]. For equations with nonsmooth kernels, we refer to Grimmer and Pritchard [19], Lunardi and Sinestrari [37], and Lorenzi and Sinestrari [36] and references therein. Furthermore finite element methods for elliptic and parabolic integro-differential equations problems with a smooth kernel have been discussed in, e.g., Cannon and Lin [5], LeRoux and Thomée [29], Lin et al. [31], Sloan and Thomée [56], Thomée and Zhang [57], and Yanik and Fairweather [66]. Very recently the finite element methods of deterministic optimal control governed by various integral-differential equations have been studied in, e.g., [23, 52, 53, 55].

Numerical method for solving stochastic partial differential equations has been an important research topic. The current main computational methods are divided into non-intrusive and intrusive methods. Non-intrusive methods such as Monte Carlo method or stochastic collocation method are based on sampling techniques resulting in some uncoupled (in terms of stochastic spaces and physical spaces) deterministic partial differential equations to be solved. The Monte Carlo method has been the most widely used non-intrusive method to simulate SPDE [6, 45]. Although MC method is straightforward to be applied and its convergence is independent of the number of stochastic dimensions, it has a slow rate of convergence, i.e., proportional to $\frac{1}{\sqrt{N}}$ where N is the number of samples [11]. Stochastic collocation method consists of a Galerkin approximation in space and a collocation of the zeros of suitable tensor product orthogonal polynomials in the probability space [1, 13, 38, 63]. It has fast convergence for problems with appropriate stochastic dimensions and smoothness. And the sparse grid stochastic collocation method is designed for the higher stochastic dimensions problem [41, 42]. Stochastic Galerkin method [17] is a representative of the intrusive methods and has been applied to various stochastic problems, see, e.g. [2, 15, 62, 64]. The disadvantage of stochastic Galerkin method is that the resulting linear system is coupled, and the advantage is that the relative smaller number of resulting in linear equations, which is only about $\frac{1}{2^p}$ (p is the order of the stochastic discretization) of the number of linear equation generated by sparse grid stochastic collocation method [63, 65]. The numerical comparisons in [3, 9] have shown that the stochastic Galerkin method has less computational cost than stochastic collocation method when efficient solvers are available. The stochastic Galerkin method allows us to use polynomial chaos or generalized polynomial chaos serving as a complete basis to represent random processes as explicit functional of finite number of independent random variables. Although the size of the coupled linear system rises sharply with the stochastic space or physical space dimensions, the matrix of this coupled linear system has particular structures which can be utilized in designing fast solvers [50]. There are a number of researches on fast iterative solvers for the stochastic Galerkin method [16, 46, 48, 59]. In [48], the block diagonal pre-conditioner (also known as mean-based pre-conditioner) for the conjugate gradient (CG) method is the most popular one that has been effectively observed for stochastic Galerkin method. For optimal control problem governed by SPDEs, there exits little work for fast solvers, such as in [4, 40]. However it is not straightforward to extend these methods to the case optimal control problem governed by SPDEs with control or state constrains, since this is nonlinear problem. For optimal control problem governed by an elliptic integro-differential equation with random coefficients, a stochastic finite element approximation scheme and the a priori error estimate for the state, the co-state and the control variables are developed in [54], although efficient computational method is still a difficult problem. To our best knowledge, there has been a lack of an efficient algorithm for the stochastic finite element approximation of any optimal control problem with control constraints.

In this work, we combine the optimality of the PPGA algorithm for the optimal control