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ANALYSIS OF MULTI-INDEX MONTE CARLO ESTIMATORS FOR A ZAKAI SPDE*

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Abstract

In this article, we propose a space-time Multi-Index Monte Carlo (MIMC) estimator for a one-dimensional parabolic stochastic partial differential equation (SPDE) of Zakai type. We compare the complexity with the Multilevel Monte Carlo (MLMC) method of Giles and Reisinger (2012), and find, by means of Fourier analysis, that the MIMC method: (i) has suboptimal complexity of $O(\varepsilon^{-2}|\log\varepsilon|^3)$ for a root mean square error (RMSE) ε if the same spatial discretisation as in the MLMC method is used; (ii) has a better complexity of $O(\varepsilon^{-2}|\log\varepsilon|)$ if a carefully adapted discretisation is used; (iii) has to be adapted for non-smooth functionals. Numerical tests confirm these findings empirically.

Mathematics subject classification: 65C05, 65T50, 60H15, 65N06, 65N12.
Key words: Parabolic stochastic partial differential equations, Multilevel Monte Carlo, Multi-index Monte Carlo, Stochastic finite differences, Zakai equation.

1. Introduction

Stochastic partial differential equations (SPDEs) have become an area of active research over the last few decades. Several classes of methods have been developed to solve SPDEs numerically, including finite difference schemes [5, 11–13], finite element schemes [19, 28], and stochastic Taylor schemes [16, 17].

This article is motivated by Zakai SPDEs of the form (see [9]),

$$\mathrm{d}v(t,x) = \left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\big[a(x)v(t,x)\big] - \frac{\partial}{\partial x}\big[b(x)v(t,x)\big]\right)\mathrm{d}t - \frac{\partial}{\partial x}\big[\gamma(x)v(t,x)\big]\,\mathrm{d}M_t, \tag{1.1}$$

where M is a standard Brownian motion and a, b and γ suitably chosen coefficient functions. This Zakai equation arises from a nonlinear filtering problem: given an observation process M and a signal process X, we want to estimate the conditional distribution of X given M. If X satisfies

$$X_t = X_0 + \int_0^t \beta(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}B_s + \int_0^t \gamma(X_s)\,\mathrm{d}M_s,$$

where B and M are independent standard Brownian motions, and the distribution function has a density v, it is proved in [21] that v satisfies (1.1) with

$$a = \sigma^2 + \gamma^2, \qquad b = \beta.$$

The conditional (on M) distribution function is then

$$L_t^x = \int_{-\infty}^x v(t,\xi)\,\mathrm{d}\xi = 1 - \int_x^\infty v(t,\xi)\,\mathrm{d}\xi,$$

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and it is the goal of this article to estimate the expectation of functionals of this form.

For simplicity, we restrict ourselves to the special case

$$\mathrm{d}v = -\mu \frac{\partial v}{\partial x}\,\mathrm{d}t + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\,\mathrm{d}t - \sqrt{\rho}\frac{\partial v}{\partial x}\,\mathrm{d}M_t, \qquad (t,x) \in (0,T) \times \mathbb{R}, \tag{1.2}$$

where $T > 0$, M is a standard Brownian motion, and μ and $0 \leq \rho < 1$ are real-valued parameters. This is a special case of (1.1) where $\sigma = \sqrt{1-\rho}$, $\gamma = \sqrt{\rho}$, $\beta = \mu$.

Moreover, v in (1.2) describes the limit empirical measure, as $N \to \infty$, of a large exchangeable particle system [21],

$$\mathrm{d}X_t^i = \mu\,\mathrm{d}t + \sqrt{1-\rho}\,\mathrm{d}W_t^i + \sqrt{\rho}\,\mathrm{d}M_t, \qquad \text{for } 1 \leq i \leq N, \tag{1.3}$$

where W_t^i and M_t are independent standard Brownian motions.

A direct application of this model is the large portfolio credit model [3]. Assume the market consists of N different firms where X_t^i are "distance-to-default" processes. Then the functional of interest is the loss

$$L_t = 1 - \int_0^\infty v(t,x)\,\mathrm{d}x, \tag{1.4}$$

i.e., the mass lost at the absorbing boundary. In the credit risk application of [2], L describes the loss in a structural credit model, i.e., the fraction of firms whose values have crossed zero and which are considered defaulted. The values of credit products are often functions of the loss L_t.

Generally, the solution to (1.1) is not known analytically and has to be approximated numerically. A survey of methods is given in [9], and we focus here on recent applications of multilevel methods as they pertain to this article. Giles and Reisinger [8] used an explicit Milstein finite difference approximation to the solution of (1.2). By using Fourier analysis, this scheme can be shown to give first order of convergence in the timestep and second order in the spatial mesh size. One constraint in this paper is that the timestep needs to be small enough to ensure stability. Inspired by the numerical analysis of SDEs in [1, 27], [25] extended the discretisation to an implicit method on the basis of the σ-θ time-stepping scheme, where the drift and the deterministic part of the double stochastic integral are taken implicit. Fourier analysis shows that the convergence order is the same as in the explicit Milstein scheme, however, this scheme is unconditionally mean-square stable under a constraint on the correlation ρ in (1.2). This unconditional stability is essential for our application as detailed below.

In this paper, we compare a new Multi-index Monte Carlo (MIMC) scheme in the spirit of [14], with the Multilevel Monte Carlo (MLMC) method of [6]. The MLMC method utilises a sequence of approximations $P_0, P_1, \ldots, P_{l^*}$ to a random variable P with increasing accuracy but also higher cost for increasing l. In the simulation of an SDE, l would typically be the refinement level of the time mesh, with 2^l time steps. The MLMC estimator is based on recursive control variates embedded in the identity

$$\mathbb{E}\left[P_{l^*}\right] = \mathbb{E}\big[P_0\big] + \sum_{l=1}^{l^*} \mathbb{E}\big[P_l - P_{l-1}\big],$$

where l^* is a maximum refinement level. The goal is to estimate $\mathbb{E}[P_{l^*}]$ by independent estimation of the summands, in a way that the root mean square error (RMSE) is comparable to the bias, but with a much reduced computational complexity. If fewer samples are needed for