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A FAST STOCHASTIC GALERKIN METHOD FOR A CONSTRAINED OPTIMAL CONTROL PROBLEM GOVERNED BY A RANDOM FRACTIONAL DIFFUSION EQUATION*

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Abstract

We develop a fast stochastic Galerkin method for an optimal control problem governed by a random space-fractional diffusion equation with deterministic constrained control. Optimal control problems governed by a fractional diffusion equation tends to provide a better description for transport or conduction processes in heterogeneous media. However, the fractional control problem introduces significant computation complexity due to the nonlocal nature of fractional differential operators, and this is further worsen by the large number of random space dimensions to discretize the probability space. We approximate the optimality system by a gradient algorithm combined with the stochastic Galerkin method through the discretization with respect to both the spatial space and the probability space. The resulting linear system can be decoupled for the random and spatial variable, and thus solved separately. A fast preconditioned Bi-Conjugate Gradient Stabilized method is developed to efficiently solve the decoupled systems derived from the fractional diffusion operators in the spatial space. Numerical experiments show the utility of the method.

Mathematics subject classification: 65C20, 65F10, 65N30, 65T50.
Key words: Constrained optimal control, Fractional diffusion, Stochastic Galerkin method, Fast Fourier transform, Preconditioned Bi-Conjugate Gradient Stabilized method.

1. Introduction

It is well known that numerical approximation of optimal control problems has long been an important topic in engineering design work. There has been extensive researches on developing fast numerical algorithms for these optimal control problems. Furthermore the optimal control problems for a classical second-order Fickian diffusion operator have been studied extensively in the literature and many efficient and reliable methods have been developed [9,21–24,27,31,35].

In the last few decades an increasingly more number of evidence suggests that the classical second-order Fickian diffusion equation does not necessarily provides a proper description of the diffusion process in a heterogeneous medium. It was shown in [5,29] that the heavy

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tail behavior of the transport processes in heterogeneous media can be described accurately by Levy distribution, which can be viewed as a probability description of fractional diffusion equations. There has been increasingly more research on both analysis and numerics for the fractional control problems [1, 13, 16]. However, because of the nonlocal nature of fractional differential operators, the corresponding numerical methods for solving the fractional equations generate full stiffness matrices. Hence, the computational bottleneck becomes a really hard issue. The first important progress in this direction was made in [42], where the authors proved that the stiffness matrix of the Meerschaert-Tadjeran method for the space-fractional diffusion equations in one space dimension has Toeplitz-like structures, and thus a fast method with a computational work acount of $O(N \log^2 N)$ was established. Later in [43] they utilized the Toeplitz-like structure of the stiffness matrix to develop a fast Krylov subspace iterative solver with a computational cost of $O(N \log N)$ per iteration. Motivated by this research, we proposed a fast gradient method for a pointwise constrained optimal control problem governed by a time-dependent space-fractional diffusion equation [14], and significantly reduce its computational complexity from $O(N^3)$ to $O(N \log N)$ for discretized problem of size N.

Uncertainty arises in many complex real-world problems of physical and engineering applications, such as variability of soil permeability in subsurface aquifers, heterogeneity of materials with microstructure, wall roughness in a fluid dynamics study, etc, and plays an increasingly important role. Based on the recent works on the numerical methods for the random PDEs ([2–4, 10, 12, 17, 32, 41, 45, 46]), it is natural to study the optimal control problem governed by PDEs with random coefficients [19, 20, 25, 36, 39, 40]. The work [19] deals with the optimal control problems for stochastic partial differential equations with Neumann boundary conditions. In [20, 25], stochastic optimal control problems constrained by stochastic elliptic PDEs with deterministic distributed control function are introduced. In [40], an optimal control problem with the deterministic control of the obstacle constraint governed by an elliptic PDE with random field is studied. However, the computation of the stochastic Galerkin method for optimal control problem is still a difficult problem since in the resulting linear system all the variables are coupled together and have to be computed as a whole, so that its computational complexity increases exponentially. For this reason, the existing algorithms [6, 30] can only deal with optimal control problems of small size.

In this paper, we develop an effective scheme to approximate the optimality system governed by a random space-fractional diffusion equation with deterministic constrained control. We approximate the optimality system by a gradient algorithm combined with the stochastic Galerkin method through the discretization with respect to both the spatial space and the probability space. The resulting linear system can be decoupled for the random and spatial variable, and thus solved separately. A fast preconditioned Bi-Conjugate Gradient Stabilized (BiCGSTAB) method is developed to efficiently solve the decoupled systems derived from the fractional diffusion operators in the spatial space. Numerical experiments show the utility of the method.

The rest of the paper is organized as follows: In Section 2, we introduce the stochastic fractional optimal control problem and derive the optimality condition. In Section 3, we represent the original problem into a finite dimensional stochastic fractional optimal control problem based on the finite dimensional noise assumption. Then the stochastic Galerkin approximation schemes combined with the gradient algorithm are presented to solve the finite dimensional optimal control problem. In Section 4, the resulting linear system is decoupled and we present