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A SPARSE GRID STOCHASTIC COLLOCATION AND FINITE VOLUME ELEMENT METHOD FOR CONSTRAINED OPTIMAL CONTROL PROBLEM GOVERNED BY RANDOM ELLIPTIC EQUATIONS*

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Abstract

In this paper, a hybird approximation scheme for an optimal control problem governed by an elliptic equation with random field in its coefficients is considered. The random coefficients are smooth in the physical space and depend on a large number of random variables in the probability space. The necessary and sufficient optimality conditions for the optimal control problem are obtained. The scheme is established to approximate the optimality system through the discretization by using finite volume element method for the spatial space and a sparse grid stochastic collocation method based on the Smolyak approximation for the probability space, respectively. This scheme naturally leads to the discrete solutions of an uncoupled deterministic problem. The existence and uniqueness of the discrete solutions are proved. A priori error estimates are derived for the state, the co-state and the control variables. Numerical examples are presented to illustrate our theoretical results.

Mathematics subject classification: 65N06, 65B99.
Key words: Optimal control problem, Random elliptic equations, Finite volume element, Sparse grid, Smolyak approximation, A priori error estimates.

1. Introduction

Optimal control problems governed by partial differential equations (PDEs) have been a major research topic in applied mathematics and control science and engineering. Computational methods for deterministic optimal control problems governed by PDEs have been well developed and investigated for several decades [12, 17–23, 30, 32, 33, 35]. Among them, the finite volume element approximation plays an important role in numerical methods for these problems. Because this method has some crucial physical conservation properties of the original problem locally, it is popular in computational fluid mechanics. There have been extensive studies on this topic, such as for elliptic equations [22] and hyperbolic equations [23].

Uncertainty, such as uncertain parameters, arises in many complex real-world problems of physical and engineering interests. It is well known that these problems can be described by different kinds of stochastic partial differential equations (SPDEs). The well-known Monte Carlo

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(MC) method is the most commonly used method for simulating SPDEs and dealing with the statistic characteristics of the solution [9,28]. Although MC method only needs to do repetitive deterministic simulations, it is a rather computationally expensive method for the reason that the statistic convergence rate is relatively slow, especialy when there are large amounts of computations in the deterministic systems. Another alternative to the Monte Carlo method is the so-called stochastic Galerkin method [11,14] for solving SPDEs with random fields input data. This method allows us to utilize standard approximations in physical space and polynomial approximation in the probability space, but in general, this technique requires to solve a system of equations that couples all degrees of freedom when approximating the stochastic systems. Due to this issue, the stochastic collocation method has gained much attention recently in the computational community [4,5,26,27,38]. Stochastic collocation method consists of a Galerkin approximation in physical space and a collocation in the zeros of suitable tensor product orthogonal polynomials (Gauss points) in the probability space [4]. Compared with stochastic Galerkin methods, this method solves uncoupled deterministic PDEs at the collocation points that are trivially parallelizable, as in the Monte Carlo method.

For optimal control problem governed by SPDEs, there have been many works based the finite element approximation for the physical space such as [31, 34]. To our best knowledge, there has been a lack of a finite volume element approximation of optimal control problem with control constraints governed by any SPDEs.

In this paper, we consider a hybird approximation scheme for an optimal control problem governed by an elliptic equation with random field in its coefficients. The random coefficients are smooth in the physical space and depend on a large number of random variables in the probability space. The plan of the paper is as follows. In Section 2, we introduce some function spaces and the stochastic optimal control problem. By applying the well-known Lions' lemma to the optimal control problem, we obtain the necessary and sufficient optimality conditions. In Section 3, we discuss the finite dimensional representation of stochastic fields, obtain the finite dimensional optimal control problem and its optimality systems. In Section 4, we develop the scheme to approximate the optimality systems by using a sparse grid stochastic collocation method based on the Smolyak approximation for the probability space and a finite volume element approximation for the physical space, and naturally leads to the discrete solutions of an uncoupled deterministic problem. In Section 5, we prove the existence and uniqueness for the discrete solutions. A priori error estimations for the stochastic optimal control problem are derived in Section 6. Finally, numerical examples for both low and high random dimensions are presented to illustrate our theoretical results in Section 7.

2. Notations and Model Control Problem

2.1. Function Spaces and Notations

Let D be a convex bounded polygonal spatial domain in $\mathbb{R}^d (1 \leq d \leq 3)$ with boundary ∂D and $B(D)$ be the Borel σ-algebra generated by the open subset of D. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where Ω is a set of outcomes, $\mathcal{F}$ is a σ-algebra of events and $P : \mathcal{F} \to [0, 1]$ is a probability measure. Let $Y = (Y_1(\omega), \cdots, Y_N(\omega))$ be an $\mathbb{R}^N-$valued random variable in $(\Omega, \mathcal{F}, P)$, and for $q \in [1, \infty)$, let $(L^q_P(\Omega))^N$ be the set comprising those random variables Y with $\sum_{i=1}^N \int_\Omega |Y_i(\omega)|^q dP(\omega) < \infty$. If $Y \in L^1_P(\Omega)$, we denote its expectation by

$$\mathbb{E}[Y] = \int_\Omega Y(\omega) dP(\omega) = \int_{\mathbb{R}^N} y d\mu_Y(y),$$