Journal of Computational Mathematics
Vol.36, No.3, 2018, 391–403.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1708-m2017-0130

ON DOUBLY POSITIVE SEMIDEFINITE PROGRAMMING RELAXATIONS*

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Abstract

Recently, researchers have been interested in studying the semidefinite programming (SDP) relaxation model, where the matrix is both positive semidefinite and entry-wise nonnegative, for quadratically constrained quadratic programming (QCQP). Comparing to the basic SDP relaxation, this doubly-positive SDP model possesses additional $O(n^2)$ constraints, which makes the SDP solution complexity substantially higher than that for the basic model with $O(n)$ constraints. In this paper, we prove that the doubly-positive SDP model is equivalent to the basic one with a set of valid quadratic cuts. When QCQP is symmetric and homogeneous (which represents many classical combinatorial and non-convex optimization problems), the doubly-positive SDP model is equivalent to the basic SDP even without any valid cut. On the other hand, the doubly-positive SDP model could help to tighten the bound up to 36%, but no more. Finally, we manage to extend some of the previous results to quartic models.

Mathematics subject classification: 90C20, 90C22, 90C26, 65K05.
Key words: Doubly nonnegative matrix, Semidefinite programming, Relaxation, quartic optimization.

1. Introduction

Consider the quadratically constrained quadratic programming problem

$$\begin{aligned} \text{Maximize} \quad & x^TQ_0x + c_0^Tx \\ \text{Subject to} \quad & x^TQ_ix + c_i^Tx = b_i, \quad i = 1, \dots, m, \\ & -e \le x \le e, \end{aligned} \tag{1.1}$$

where symmetric matrix $Q_i \in \Re^{n\times n}$ and vector $c_i \in \Re^n$, $i = 0, 1, \cdots, m$, and $e \in \Re^n$ is the vector of all ones. Note that any other lower and upper bounds on decision variables, $l \le x \le u$, can be transformed to $-e \le x \le e$ through scaling and linear translation. Also, the results developed in this paper are easily extendable to quadratic inequality constraints. We assume

* Received May 23, 2017 / Revised version received June 12, 2017 / Accepted August 4, 2017 / Published online March 28, 2018 /

that the QP problem is known to be feasible, so that any of its relaxation models would be also feasible.

The classical and basic semidefinite programming relaxation for problem (1.1) is

$$\begin{aligned} \text{Maximize} \quad & Q_0 \cdot X + c_0^T x \\ \text{Subject to} \quad & Q_i \cdot X + c_i^T x = b_i, \quad i = 1, \cdots, m, \\ & X_{jj} \le 1, \quad j = 1, \cdots, n, \\ & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0. \end{aligned} \tag{1.2}$$

If the SDP solution has a rank one property, that is, $X^* = x^*(x^*)^T$, then x^* solves problem (1.1).

Recently, there are research efforts to construct stronger or tighter SDP relaxations for QCQP. One particular effort is to let $y = (x + e)/2$ so that problem (1.1) has an equivalent form within the nonnegative domain:

$$\begin{aligned} \text{Maximize} \quad & 4y^T Q_0 y + (2c_0^T - 4e^T Q_0)y + e^T Q_0 e - c_0^T e \\ \text{Subject to} \quad & 4y^T Q_i y + (2c_i^T - 4e^T Q_i)y + e^T Q_i e - c_i^T e = b_i, \quad i = 1, \ldots, m, \\ & 0 \le y \le e. \end{aligned} \tag{1.3}$$

Using the knowledge that all decision variables need to be nonnegative, the following SDP relaxation can be constructed:

$$\begin{aligned} v_p^* := \quad \text{Maximize} \quad & 4Q_0 \cdot Y + (2c_0^T - 4e^T Q_0)y + e^T Q_0 e - c_0^T e \\ \text{Subject to} \quad & 4Q_i \cdot Y + (2c_i^T - 4e^T Q_i)y + e^T Q_i e - c_i^T e = b_i, \quad i = 1, \ldots, m, \\ & Y_{jj} \le y_j \le 1, \ j = 1, 2, \cdots, n, \\ & Z := \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \succeq 0, \\ & Y_{ij} \ge 0, \ \forall 1 \le i < j \le n. \end{aligned} \tag{1.4}$$

Since $Z \succeq 0$ as well as $Z \ge 0$, it is called a *doubly-positive semidefinite program*; e.g., see Dong et al. [5], Burer [3] and Burer et al. [4]. It is well known that there exists a hierarchy of linear and semidefinite representable cones that approximate the co-positive and completely positive cone (see Bomze et al. [2] and Parrilo [13]), where the doubly-positive SDP is a mostly used relaxation technique due to its computability. Very recent research has discussed its applications in many areas, e.g., appointment scheduling by Kong et al. [10], order statistics by Natarajan et al. [11].

The doubly-positive SDP increases the number of constraints from $m + n$ in basic SDP model (1.2) to $m + 2n + n(n-1)/2$ in (1.4). With such a sacrifice in computational complexity, (1.4) must be stronger or tighter than (1.2). In this paper, we are trying to answer this very question: when and how much is the doubly-positive SDP relaxation tighter than the basic SDP one?

Besides, in the last section, we manage to extend the doubly-positive relaxation to the quartic optimization, which has wide applications in sensor network localization [1], portfolio management with high moments information [9] and et. al., and obtain some results which are similar to the quadratic optimization.

2. The Basic SDP Relaxation with Valid Cuts

In fact, one can add valid cuts

$$1 + X_{ij} + x_i + x_j \geq 0, \quad \forall 1 \leq i < j \leq n;$$

into basic model (1.2) from the fact that binary product $(1+x_i)(1+x_j) \geq 0$ always in original problem (1.1). Then, SDP relaxation model (1.2) becomes

$$\begin{aligned} v_s^* := \quad & \text{Maximize} \quad & & Q_0 \cdot X + c_0^T x \\ & \text{Subject to} \quad & & Q_i \cdot X + c_i^T x = b_i, \quad i = 1, \cdots, m, \\ & & & X_{jj} \leq 1, \quad j = 1, \cdots, n, \\ & & & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0, \\ & & & 1 + X_{ij} + x_i + x_j \geq 0, \ \forall 1 \leq i < j \leq n. \end{aligned} \tag{2.1}$$

This strengthened SDP relaxation has $m + n + n(n-1)/2$ constraints. We now show the following equivalence theorem:

Theorem 2.1. *SDP relaxations* (2.1) *and* (1.4) *produce exactly the same optimal objective value, that is,* $v_s^* = v_p^*$.

Proof. First, we prove $v_p^* \geq v_s^*$.

Given an optimal solution (X^*, x^*) to (2.1), let $Y_{ij} = \frac{1}{4}(X_{ij}^* + x_i^* + x_j^* + 1)$ and $y_j = (x_j^* + 1)/2$, $1 \leq i, j \leq n$, that is, $Y = \frac{1}{4}(X^* + e(x^*)^T + x^* e^T + ee^T)$ and $y = (x^* + e)/2$.

First, one can see, for all $i \leq j \leq n$, $Y_{ij} \geq 0$, since $X_{ij}^* + x_i^* + x_j^* + 1 \geq 0$ in (2.1). Secondly,

$$\begin{aligned} Y - yy^T = & \frac{1}{4}(X^* + e(x^*)^T + x^* e^T + ee^T) - \frac{1}{4}(x^* + e)(x^* + e)^T \\ = & \frac{1}{4}(X^* - x^*(x^*)^T) \succeq 0, \end{aligned}$$

so that $Z \succeq 0$. Thirdly,

$$Y_{jj} = \frac{1}{4}(X_{jj}^* + 2x_j^* + 1) \leq (2 + 2x_j^*)/4 = (1 + x_j^*)/2 = y_j, \quad \forall \ j = 1, \cdots, n.$$

Finally, for $i = 0, 1, \cdots, n$,

$$\begin{aligned} & 4Q_i \cdot Y + (2c_i^T - 4e^T Q_i)y + e^T Q_i e - c_i^T e \\ = \ & Q_i \cdot (X^* + e(x^*)^T + x^* e^T + ee^T) + (c_i^T - 2e^T Q_i)(x^* + e) + e^T Q_i e - c_i^T e \\ = \ & Q_i \cdot X^* + c_i^T x^*. \end{aligned}$$

Thus (Y, y) is a feasible solution to (1.4) and its objective value equals v_s^*, which implies $v_p^* \geq v_s^*$.

Next we prove $v_p^* \leq v_s^*$. Given an optimal solution (Y^*, y^*) to (1.4), we let $X_{ij} = 4Y_{ij}^* - 2y_i^* - 2y_j^* + 1$ and $x_j = 2y_j^* - 1$, $1 \leq i, j \leq n$, that is, $X = 4Y^* - 2e(y^*)^T - 2y^* e^T + ee^T$ and $x = 2y^* - e$.

First, one can see, for all $i \leq j \leq n$,

$$X_{ij} + x_i + x_j + 1 = 4Y_{ij}^* \geq 0;$$

and for all $j = 1, \cdots, n$, $X_{jj} = 4(Y^*_{jj} - y^*_j) + 1 \leq 1$. Secondly,

$$\begin{aligned} X - xx^T &= 4Y^* - 2e(y^*)^T - 2y^*e^T + ee^T - (2y^* - e)(2y^* - e)^T \\ &= 4(Y^* - y^*(y^*)^T) \succeq 0. \end{aligned}$$

Finally, for $i = 0, 1, \cdots, n$,

$$\begin{aligned} & Q_i \cdot X + c_i^T x \\ = \quad & Q_i \cdot \left(4Y^* - 2e(y^*)^T - 2y^*e^T + ee^T\right) + c_i^T(2y^* - e) \\ = \quad & 4Q_i \cdot Y^* + (2c_i^T - 4e^TQ_i)y^* + e^TQ_ie - c_i^Te. \end{aligned}$$

Thus X is a feasible solution to (2.1) whose objective value equals v_p^*, which implies $v_p^* \leq v_s^*$. This completes the proof of the theorem. □

Theorem 2.1 implies that the doubly-positive SDP relaxation is precisely the basic SDP relaxation with additional $n(n-1)/2$ valid binary product cuts. (In fact, the latter has n constraints fewer than the former does.) It also has practical implication, since an effective rounding procedure have been developed and analyzed for basic model (1.2), which can be extended to strengthened relaxation model (2.1). To the best our knowledge, there is no effective rounding procedure currently available for SDP relaxation model (1.4).

3. Homogeneous and Symmetric QCQP

In this section, we show that the doubly-positive SDP relaxation is precisely the basic SDP relaxation even without the valid binary quadratic cuts, when the quadratic functions are homogeneous. Here we consider the homogeneous QCQP problem:

$$\begin{aligned} \text{Maximize} \quad & x^TQ_0x \\ \text{Subject to} \quad & x^TQ_ix = b_i, \quad i = 1, \ldots, m, \\ & -e \leq x \leq e. \end{aligned} \tag{3.1}$$

The solution set of (3.1) is symmetric, meaning that when x is a solution, so is $-x$.

In general, problem (3.1) remains NP-hard, and it includes many classical combinatorial and non-convex optimization problems. Since there is no linear term in each of the quadratic function, the basic SDP relaxation can be simplified to:

$$\begin{aligned} v_b^* := \quad \text{Maximize} \quad & Q_0 \cdot X \\ \text{Subject to} \quad & Q_i \cdot X = b_i, \quad i = 1, \cdots, m, \\ & X_{jj} \leq 1, \quad j = 1, \cdots, n, \\ & X \succeq 0. \end{aligned} \tag{3.2}$$

Note that basic relaxation (3.2) has only $m + n$ constraints.

If (3.1) represents the max-cut problem, Goemans and Williamson [6] developed a rounding procedure to produce a feasible solution to (3.1) from the maximal solution matrix of (3.2) such that the feasible solution is 0.878-optimal. If (3.1) represents the max-bisection problem, Ye [17] developed a rounding procedure to produce a feasible solution that is 0.699-optimal.

More generally, when coefficient matrix Q_i is diagonal for all i, Nesterov [12] and Ye [16] developed a rounding procedure to produce a feasible solution that is $\frac{2}{\pi}$-optimal.

Note now that the doubly-positive SDP relaxation of (3.1) becomes

$$\begin{aligned} v_p^* := \quad & \text{Maximize} \quad 4Q_0 \cdot Y - 4e^T Q_0 y + e^T Q_0 e \\ & \text{Subject to} \quad 4Q_i \cdot Y - 4e^T Q_i y + e^T Q_i e = b_i, \quad i = 1, \cdots, m, \\ & \qquad Y_{jj} \le y_j \le 1, \quad j = 1, \cdots, n, \\ & \qquad Z = \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \succeq 0, \\ & \qquad Z \ge 0. \end{aligned} \tag{3.3}$$

We prove that, in the homogeneous and symmetric case, the doubly-positive SDP relaxation adds no value to the basic relaxation:

Theorem 3.1. *SDP relaxations* (3.2) *and* (3.3) *produce exactly the same optimal objective value, that is,* $v_b^* = v_p^*$.

Proof. The proof is similar to the proof of Theorem 2.1 by treading $x^* = 0$.

Given an optimal solution X^* to (3.2), let $Y_{ij} = \frac{1}{4}(X^*_{ij} + 1)$ and $y_j = \frac{1}{2}$, $1 \le i, j \le n$, that is, $Y = \frac{1}{4}(X^* + ee^T)$ and $y = e/2$.

First, one can see $Y_{ij} \ge 0$ since $-1 \le X^*_{ij} \le 1$ for all i, j. Secondly,

$$Y - yy^T = \frac{1}{4}(X^* + ee^T) - \frac{1}{4}ee^T = \frac{1}{4}X^* \succeq 0.$$

Thirdly, $Y_{jj} = \frac{1}{4}(X^*_{jj} + 1) \le 1/2 = y_j$. Finally, for $i = 0, 1, \cdots, n$,

$$\begin{aligned} & 4Q_i \cdot Y - 4e^T Q_i y + e^T Q_i e \\ = {} & Q_i \cdot (X^* + ee^T) - 2Q_i \cdot ee^T + Q_i \cdot ee^T \\ = {} & Q_i \cdot X^*. \end{aligned}$$

Thus (Y, y) is a feasible solution to (3.3) and its objective value equals to v_b^*. This implies $v_p^* \ge v_b^*$.

On the other hand, given an optimal solution (Y^*, y^*) to (3.3), we let

$$X_{ij} = 4Y^*_{ij} - 2y^*_i - 2y^*_j + 1.$$

First, from $X_{ij} = 4(Y^*_{ij} - y^*_i y^*_j) + 4y^*_i y^*_j - 2y^*_i - 2y^*_j + 1$, we see

$$X = 4(Y^* - y^*(y^*)^T) + (2y^* - e)(2y^* - e)^T \succeq 0.$$

Secondly, we have $X_{jj} = 4Y^*_{jj} - 4y^*_j + 1 \le 1$. Finally, for $i = 0, 1, \cdots, n$,

$$\begin{aligned} & Q_i \cdot X \\ = {} & Q_i \cdot \left(4(Y^* - y^*(y^*)^T) + (2y^* - e)(2y^* - e)^T\right) \\ = {} & 4Q_i \cdot Y^* - 4Q_i \cdot yy^T + 4Q_i \cdot y^*(y^*)^T - 4e^T Q_i y^* + Q_i \cdot ee^T \\ = {} & 4Q_i \cdot Y^* - 4e^T Q_i y^* + e^T Q_i e. \end{aligned}$$

Thus X is a feasible solution to (3.2) whose objective value equals to v_p^*. This implies that $v_p^* \le v_b^*$. □

4. How Tighter the Doubly-Positive SDP Model is

Now we ask the question: what difference the doubly-positive SDP relaxation could make comparing to the basic one without any valid cut. Due to Theorem 2.1, the question is as the same as: what difference strengthened SDP relaxation (2.1) could make comparing to basic model (1.2). For simplicity, we consider SDP relaxation for inhomogeneous binary QP:

$$\begin{aligned} v_b^* := \quad & \text{Maximize} \quad Q \cdot \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \\ & \text{Subject to} \quad X_{jj} = 1, \quad j = 1, \cdots, n, \\ & \qquad\qquad \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0, \end{aligned} \tag{4.1}$$

where the coefficient matrix $Q \in \Re^{(n+1)\times(n+1)}$.

The strengthened SDP relaxation model becomes

$$\begin{aligned} v_p^* := \quad & \text{Maximize} \quad Q \cdot \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \\ & \text{Subject to} \quad Y_{jj} = 1, \quad j = 1, \cdots, n, \\ & \qquad\qquad \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \succeq 0, \\ & \qquad\qquad 1 + Y_{ij} + y_i + y_j \geq 0, \quad \forall\ 1 \leq i < j \leq n. \end{aligned} \tag{4.2}$$

Our next theorem is

Theorem 4.1. *For SDP problems* (4.1) *and* (4.2), *if Q is positive semidefinite then*

$$v_b^* \geq v_p^* \geq \frac{2}{\pi} v_b^*;$$

if Q is a graph Laplacian matrix with nonnegative edge weights then

$$v_b^* \geq v_p^* \geq 0.878 v_b^*.$$

Proof. It is clear that $v_b^* \geq v_p^*$. Let (X^*, x^*) be a maximal solution of SDP (4.1), and let

$$Y_{ij} = \frac{2}{\pi}\arcsin(X_{ij}^*) \quad \text{and} \quad y_j = \frac{2}{\pi}\arcsin(x_j^*)$$

for all $1 \leq i \leq j \leq n$. That is,

$$\begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} = \frac{2}{\pi}\arcsin\left[\begin{bmatrix} X^* & x^* \\ (x^*)^T & 1 \end{bmatrix}\right].$$

Since for all $1 \leq i \leq j \leq n$,

$$\begin{bmatrix} 1 & X_{ij}^* & x_i^* \\ X_{ij}^* & 1 & x_j^* \\ x_i^* & x_j^* & 1 \end{bmatrix} \succeq 0,$$

we claim

$$1 + \frac{2}{\pi}\Big(\arcsin(X_{ij}^*) + \arcsin(x_i^*) + \arcsin(x_j^*)\Big) \geq 0.$$

This is because that, when we let

$$\hat{x}_j = \begin{cases} 1 & \text{if } u_j \geq 0, \\ -1 & \text{if } u_j < 0; \end{cases}$$

where u is a multivariate normal random vector with 0 mean and the covariance matrix $\begin{bmatrix} X^* & x^* \\ (x^*)^T & 1 \end{bmatrix}$, then let $\hat{x}_j = \hat{x}_{n+1}\hat{x}_j$ for $1 \leq j \leq n$; we have always

$$1 + \hat{x}_i + \hat{x}_j + \hat{x}_i\hat{x}_j \geq 0,$$

so that

$$\mathrm{E}[1 + \hat{x}_i\hat{x}_j + \hat{x}_i + \hat{x}_j] \geq 0.$$

Furthermore,

$$\mathrm{E}[1 + \hat{x}_i\hat{x}_j + \hat{x}_i + \hat{x}_j] = 1 + \frac{2}{\pi}\Big(\arcsin(X^*_{ij}) + \arcsin(x^*_i) + \arcsin(x^*_j)\Big)$$

from Sheppard [15], which proves the claim.

Thus, (Y, y) is a feasible solution to SDP (4.2), so that

$$v^*_p \geq Q \cdot \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} = Q \cdot \frac{2}{\pi} \arcsin\left[\begin{bmatrix} X^* & x^* \\ (x^*)^T & 1 \end{bmatrix}\right] \geq \frac{2}{\pi} Q \cdot \begin{bmatrix} X^* & x^* \\ (x^*)^T & 1 \end{bmatrix} = \frac{2}{\pi} v^*_b.$$

Here, we have used a fact that $\arcsin[X] \succeq X$ when X is a feasible solution to (4.1); see Nesterov [12].

If Q is a Laplacian matrix with nonnegative weights, Goemans and Williamson [6] have showed that

$$Q \cdot \frac{2}{\pi} \arcsin\left[\begin{bmatrix} X^* & x^* \\ (x^*)^T & 1 \end{bmatrix}\right] \geq 0.878 Q \cdot \begin{bmatrix} X^* & x^* \\ (x^*)^T & 1 \end{bmatrix} = 0.878 v^*_b.$$

□

Below is a small example to show the tightness of the bound. Consider an example where the Laplacian matrix

$$Q = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

in SDPs (4.1) and (4.2). One can see that $v^*_b = 9$ and the opimal solution of SDP (4.1) is:

$$\begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix},$$

$v^*_p = 8$ and the optimal solution of SDP (4.2) is:

$$\begin{bmatrix} 1 & -0.6 & -0.2 \\ -0.6 & 1 & -0.2 \\ -0.2 & -0.2 & 1 \end{bmatrix}.$$

We remark that Theorem 4.1 applies to a more strengthened SDP model where more valid (triangle) quadratic constraints

$$1 + X_{ij} + X_{jk} + X_{ik} \geq 0$$

into (1.2) when original decision variables are either 1 or -1; or

$$2 + Y_{ii} + Y_{jj} + Y_{kk} + 2(Y_{ij} + Y_{jk} + Y_{ik}) - 3(y_i + y_j + y_k) \geq 0$$

are added into (1.4) when original decision variables are either 1 or 0, for all $1 \leq i < j < k \leq n$. These cuts actually have $O(n^3)$ many.

We also remark that Theorem 4.1 applies to general SDP (1.2) where Q_i is diagonal and $c_i = 0$ for $i = 1, \cdots, m$ described in [16], that is, the doubly-positive SDP model, or the strengthened SDP model, could help to tighten the SDP upper bound up to $1 - \frac{2}{\pi}$, or about 36%, but no more.

5. Extensions to Quartic Optimization

In this section, we extend our results to quartic optimization.

5.1. Homogeneous quartic optimization

First we consider a homogeneous case. We denote a homogeneous quartic polynomial by

$$\mathcal{F}(x, x, x, x) := \sum_{i,j,k,\ell=1}^{n} \mathcal{F}_{ijk\ell} x_i x_j x_k x_\ell,$$

where $\mathcal{F} \in \Re^{n^4}$ is a fourth order symmetric tensor, and its entries are invariant under any permutation of indices. Consider the following quartic optimization problem

$$\begin{array}{ll} \text{Maximize} & \mathcal{F}(x, x, x, x) \\ \text{Subject to} & \mathcal{G}_i(x, x, x, x) = b_i, \ i = 1, \ldots, m, \\ & -e \leq x \leq e, \end{array} \tag{5.1}$$

with $\mathcal{F}$, $\mathcal{G}_i$, $i = 1, \ldots, m$ being the fourth order symmetric tensors. To construct a tensor analogy of doubly-positive SDP, recall that a positive semidefinite matrix $A \in \Re^{n \times n}$ can be decomposed as

$$A = \sum_{j=1}^{m} (v^j)^{\otimes 2} := \sum_{j=1}^{m} v^j \otimes v^j$$

for some vectors $v^1, ..., v^m \in \Re^n$, where '$\otimes$' represents the outer product of vectors. In fact, such decomposition has been extended to fourth-order tensor in [7].

Definition 5.1. ([7]) *A symmetric fourth-order tensor $\mathcal{F} \in \Re^{n^4}$ is called SOP (sum of powers) if there exist m vectors $v^1, ..., v^m \in \Re^n$ such that*

$$\mathcal{F} = \sum_{i=1}^{m} (v^i)^{\otimes 4} := \sum_{i=1}^{m} v^i \otimes v^i \otimes v^i \otimes v^i.$$

The set of SOP tensors is denoted by $\Sigma^4_{n,4}$.

We further denote "$\cdot$" to be the inner product between tensors, and the following equality holds

$$\mathcal{F} \cdot \mathcal{G} = \sum_{i,j,k,\ell=1}^{n} \mathcal{F}_{ijk\ell} \mathcal{G}_{ijk\ell},$$

for any given $\mathcal{F}, \mathcal{G} \in \Re^{n^4}$. Thus we have $\mathcal{F}(x,x,x,x) = \mathcal{F}\cdot(x\otimes x\otimes x\otimes x)$. Denote $\mathcal{X} = x\otimes x\otimes x\otimes x$ in (5.1). With Definition 5.1, we have:

$$\begin{aligned}
\text{Maximize} \quad & \mathcal{F}\cdot\mathcal{X} \\
\text{Subject to} \quad & \mathcal{G}_i\cdot\mathcal{X} = b_i, \quad i = 1,\dots,m, \\
& \mathcal{X}_{j,j,j,j} \le 1, \quad j = 1,\dots,n, \\
& \mathcal{X} \in \Sigma^4_{n,4}, \\
& rank(\mathcal{X}) = 1,
\end{aligned}$$

where $rank(\mathcal{X}) = 1$ is equivalent to that there exists a vector $z \in \Re^n$ such that $\mathcal{X} = z\otimes z\otimes z\otimes z$. By dropping the rank-one constraint, we obtain a basic relaxation

$$\begin{aligned}
v_b^* = \quad \text{Maximize} \quad & \mathcal{F}\cdot\mathcal{X} \\
\text{Subject to} \quad & \mathcal{G}_i\cdot\mathcal{X} = b_i, \quad i = 1,\dots,m, \\
& \mathcal{X}_{j,j,j,j} \le 1, \quad j = 1,\dots,n, \\
& \mathcal{X} \in \Sigma^4_{n,4}.
\end{aligned} \tag{5.2}$$

To obtain doubly-positive tensor relaxation, we define the doubly-positive fourth-order tensor as follows.

Definition 5.2. *$\mathcal{F}$ is called doubly-positive if all of its entries are nonnegative and it is SOP.*

Denote $y = (x+e)/2$ and we have another equivalent form of problem (5.1) with the nonnegative constraint:

$$\begin{aligned}
\text{Maximize} \quad & \mathcal{F}(2y-e, 2y-e, 2y-e, 2y-e) \\
\text{Subject to} \quad & \mathcal{G}_i(2y-e, 2y-e, 2y-e, 2y-e) = b_i, \quad i = 1,\dots,m, \\
& 0 \le y \le e.
\end{aligned}$$

Note that for $j = 1,\dots,n$, $0 \le y_j \le 1$ is equivalent to $y_j(y_j-1) \le 0$, which is further equivalent to a quartic constraint

$$2y_j^4 - 4y_j^3 + 3y_j^2 - y_j = y_j(y_j-1)(2y_j^2 - 2y_j + 1) \le 0,$$

since $2y_j^2 - 2y_j + 1 = y_j^2 + (y_j-1)^2 > 0$. Combining those facts with Definition 5.2, the doubly-positive relaxation of problem (5.1) can be constructed as:

$$\begin{aligned}
v_d^* = \quad \text{Maximize} \quad & 16\mathcal{F}\cdot\mathcal{Y} - 32\mathcal{F}\cdot(e\otimes y^{\otimes 3}) + 24\mathcal{F}\cdot(e^{\otimes 2}\otimes y^{\otimes 2}) - 8\mathcal{F}\cdot(e^{\otimes 3}\otimes y) + \mathcal{F}\cdot(e^{\otimes 4}) \\
\text{Subject to} \quad & 16\mathcal{G}_i\cdot\mathcal{Y} - 32\mathcal{G}_i\cdot(e\otimes y^{\otimes 3}) + 24\mathcal{G}_i\cdot(e^{\otimes 2}\otimes y^{\otimes 2}) - 8\mathcal{G}_i\cdot(e^{\otimes 3}\otimes y) \\
& \quad +\mathcal{G}_i\cdot(e^{\otimes 4}) = b_i, \quad i = 1,\dots,m, \\
& y_j \le 1, \quad j = 1,\dots,n, \\
& \mathcal{Y}_{j,j,j,j} \le 2(y_j)^3 - \tfrac{3}{2}(y_j)^2 + \tfrac{1}{2}y_j, \quad j = 1,\dots,n, \\
& \mathcal{Y} - y^{\otimes 4} \in \Sigma^4_{n,4}, \\
& \mathcal{Y} \ge 0.
\end{aligned} \tag{5.3}$$

The inequality constraints in (5.3) are obtained by relaxation. Now we can present the main result of this subsection, which states the doubly-positive relaxation for homogeneous quartic polynomial problem (5.1) is precisely the basic relaxation without any additional constraints, and is similar to Theorem 3.1.

Proposition 5.1. *Relaxation* (5.2) *and* (5.3) *produce exactly the same optimal value.*

Proof. Given an optimal solution $\mathcal{X}^*$ to (5.2), let $\mathcal{Y} = \frac{1}{16}(\mathcal{X}^* + e^{\otimes 4})$ and $y = e/2$. In the following, we shall show that $(\mathcal{Y}, y)$ is a feasible solution to (5.3).

For the first constraint, we have

$$\begin{aligned}
&16\mathcal{G}_i \cdot \mathcal{Y} - 32\mathcal{G}_i \cdot (e_n \otimes y^{\otimes 3}) + 24\mathcal{G}_i \cdot (e^{\otimes 2} \otimes y^{\otimes 2}) - 8\mathcal{G}_i \cdot (e^{\otimes 3} \otimes y) + \mathcal{G}_i \cdot (e^{\otimes 4})\\
&= \mathcal{G}_i \cdot (\mathcal{X}^* + e^{\otimes 4}) - 4\mathcal{G}_i \cdot e^{\otimes 4} + 6\mathcal{G}_i \cdot e^{\otimes 4} - 4\mathcal{G}_i \cdot e^{\otimes 4} + \mathcal{G}_i \cdot e^{\otimes 4}\\
&= \mathcal{G}_i \cdot \mathcal{X}^* = b_i,
\end{aligned}$$

for $i = 1, \ldots, m$. For the second constraint,

$$\mathcal{Y} - y^{\otimes 4} = \frac{1}{16}(\mathcal{X}^* + e^{\otimes 4}) - \frac{1}{16}e^{\otimes 4} = \frac{1}{16}\mathcal{X}^* \in \Sigma^4_{n,4}.$$

The third constraint $y_j \le 1,\ j = 1, \ldots, n$ is trivial. For the remaining constraints, we have that $-1 \le \mathcal{X}^*_{i,j,k,\ell} \le 1$ for all $1 \le i, j, k, \ell \le n$, and therefore $\mathcal{Y} \ge 0$. Since $\mathcal{X}^* \in \Sigma^4_{n,4}$, there exists $v_s \in \Re^n$ for $s = 1, \ldots, m$, such that $\mathcal{X}^* = \sum_{s=1}^m v_s^{\otimes 4}$. We have

$$\begin{aligned}
|\mathcal{X}^*_{i,j,k,\ell}| = \left|\sum_{s=1}^m v_{s,i}v_{s,j}v_{s,k}v_{s,\ell}\right| &\le \frac{1}{4}\sum_{s=1}^m \left(v_{s,i}^2 + v_{s,j}^2\right)\left(v_{s,k}^2 + v_{s,\ell}^2\right)\\
&\le \frac{1}{4}\sum_{s=1}^m \left(v_{s,i}^4 + v_{s,j}^4 + v_{s,k}^4 + v_{s,\ell}^4\right)\\
&= \frac{1}{4}\left(\mathcal{X}^*_{i,i,i,i} + \mathcal{X}^*_{j,j,j,j} + \mathcal{X}^*_{k,k,k,k} + \mathcal{X}^*_{\ell,\ell,\ell,\ell}\right) \le 1.
\end{aligned}$$

Hence

$$\mathcal{Y}_{j,j,j,j} = \frac{1}{16}(\mathcal{X}^*_{j,j,j,j} + 1) \le \frac{1}{8} = 2(y_j)^3 - \frac{3}{2}(y_j)^2 + \frac{1}{2}y_j, \quad j = 1, \ldots, n,$$

and we prove that $(\mathcal{Y}, y)$ is a feasible solution.

Now let us calculate the corresponding value of the objective function

$$\begin{aligned}
&16\mathcal{F} \cdot \mathcal{Y} - 32\mathcal{F} \cdot (e \otimes y^{\otimes 3}) + 24\mathcal{F} \cdot (e^{\otimes 2} \otimes y^{\otimes 2}) - 8\mathcal{F} \cdot (e^{\otimes 3} \otimes y) + \mathcal{F} \cdot (e^{\otimes 4})\\
&= \mathcal{F} \cdot \mathcal{X}^* = v_b^*.
\end{aligned}$$

In summary, $(\mathcal{Y}, y)$ is a feasible solution to (5.3) with objective function value v_b^*, which implies that the optimal value of (5.3) $v_d^* \ge v_b^*$.

On the other hand, if $(\mathcal{Y}^*, y^*)$ is optimal to (5.3), let

$$\mathcal{X} = 16\mathcal{Y}^* - 32e \otimes y^{*\otimes 3} + 24e^{\otimes 2} \otimes y^{*\otimes 2} - 8e^{\otimes 3} \otimes y^* + e^{\otimes 4}.$$

Note that $\mathcal{X} = 16(\mathcal{Y}^* - y^{*\otimes 4}) + (2y^* - e)^{\otimes 4} \in \Sigma^4_{n,4}$,

$$\begin{aligned}\mathcal{X}_{j,j,j,j} &= 16\mathcal{Y}^*_{j,j,j,j} - 32(y^*_j)^3 + 24(y^*_j)^2 - 8y^*_j + 1\\ &= 16\left(\mathcal{Y}^*_{j,j,j,j} - 2(y^*_j)^3 + \frac{3}{2}(y^*_j)^2 - \frac{1}{2}y^*_j\right) + 1 \le 1,\end{aligned}$$

for $j = 1, \ldots, n$, and

$$\mathcal{G}_i \cdot \mathcal{X} = \mathcal{G}_i \cdot (16\mathcal{Y}^* - 32e \otimes y^{*\otimes 3} + 24e^{\otimes 2} \otimes y^{*\otimes 2} - 8e^{\otimes 3} \otimes y^* + e^{\otimes 4}) = b_i,$$

for $i = 1, \ldots, m$. Moreover, the corresponding objective function value is $\mathcal{F} \cdot \mathcal{X} = v^*_d$. Therefore, we have constructed a feasible solution $\mathcal{X}$ of problem (5.2) that has objective function value v^*_d, which implies that the optimal value of (5.2) $v^*_b \ge v^*_d$, and the proof is completed. □

5.2. Inhomogeneous quartic optimization

In this subsection, we consider a specific inhomogenous quartic problem:

$$\begin{array}{ll}\text{Maximize} & \mathcal{F}(x,x,x,x) + x^T Qx + c\\ \text{Subject to} & x_i \in \{-1,1\}, \quad i = 1, \ldots, n,\end{array} \tag{5.4}$$

where $\mathcal{F} \in \Re^{n^4}$ is a symmetric tensor and $Q \in \Re^{n\times n}$ is a symmetric matrix. Similar to previous discussion, problem (5.4) can be rewritten as

$$\begin{array}{ll}\text{Maximize} & \mathcal{F} \cdot \mathcal{X} + x^T Qx + c\\ \text{Subject to} & x_i \in \{-1,1\}, \quad i = 1, \ldots, n,\\ & \mathcal{X} - x^{\otimes 4} \in \Sigma^4_{n,4},\\ & rank(\mathcal{X}) = 1.\end{array} \tag{5.5}$$

We can further rewrite the problem as a matrix optimization problem. To be specific, for a given tensor $\mathcal{F} \in \Re^{n^4}$, its square matricization, denoted as $\mathbf{M}(\mathcal{F})$, can be written as

$$\mathbf{M}(\mathcal{F})_{ts} = \mathcal{F}_{i,j,k,\ell} \quad 1 \le i,j,k,\ell \le n$$

where $t = (i-1)n + j$ and $s = (k-1)n + \ell$. And the vectorization of matrix $A \in \Re^{n\times n}$, denoted as $\mathbf{V}(A)$, is given by

$$\mathbf{V}(A)_k = \mathcal{F}_{i,j} \quad 1 \le i,j \le n,$$

where $k = (i-1)n + j$. Now it's easy to verify that problem (5.5) is equivalent to

$$\begin{array}{ll}\text{Maximize} & P \cdot \begin{bmatrix}\mathbf{M}(\mathcal{X}) & \mathbf{V}(xx^T)\\ \left(\mathbf{V}(xx^T)\right)^T & 1\end{bmatrix}\\ \text{Subject to} & \mathbf{V}(xx^T)_k = 1,\ k \in S = \{1, n+2, 2n+3, \ldots, n^2\}\\ & \mathbf{V}(xx^T)_k \in \{-1,1\},\ k \in \{1, \ldots, n^2\} \setminus S,\\ & \begin{bmatrix}\mathbf{M}(\mathcal{X}) & \mathbf{V}(xx^T)\\ \left(\mathbf{V}(xx^T)\right)^T & 1\end{bmatrix} \succeq 0,\\ & rank(\mathbf{M}(\mathcal{X})) = 1,\end{array} \tag{5.6}$$

where

$$P = \begin{bmatrix} \mathbf{M}(\mathcal{F}) & \frac{\mathbf{V}(Q)}{2} \\ \left(\frac{\mathbf{V}(Q)}{2}\right)^T & c \end{bmatrix} \in \Re^{(n^2+1)\times(n^2+1)},$$

and the rank-one equivalence is due to [8]. By dropping the rank-one constraint, we can get a basic relaxation:

$$\begin{aligned} \text{Maximize} \quad & P \cdot \begin{bmatrix} \mathbf{M}(\mathcal{X}) & \mathbf{V}(xx^T) \\ \left(\mathbf{V}(xx^T)\right)^T & 1 \end{bmatrix} \\ \text{Subject to} \quad & \mathbf{M}(\mathcal{X})_{kk} = 1, \ k = 1, \dots n^2, \\ & \begin{bmatrix} \mathbf{M}(\mathcal{X}) & \mathbf{V}(xx^T) \\ \left(\mathbf{V}(xx^T)\right)^T & 1 \end{bmatrix} \succeq 0. \end{aligned} \tag{5.7}$$

Similar to the quadratic model, we can produce a strengthened relaxation as follows,

$$\begin{aligned} \text{Maximize} \quad & P \cdot \begin{bmatrix} \mathbf{M}(\mathcal{X}) & \mathbf{V}(xx^T) \\ [1ex] \left(\mathbf{V}(xx^T)\right)^T & 1 \end{bmatrix} \\ \text{Subject to} \quad & \mathbf{M}(\mathcal{X})_{kk} = 1, \ k = 1, \dots n^2, \\ & \begin{bmatrix} \mathbf{M}(\mathcal{X}) & \mathbf{V}(xx^T) \\ [1ex] \left(\mathbf{V}(xx^T)\right)^T & 1 \end{bmatrix} \succeq 0, \\ & 1 + \mathbf{M}(\mathcal{X})_{ij} + \mathbf{V}(xx^T)_i + \mathbf{V}(xx^T)_j \geq 0, \quad \forall 1 \leq i < j \leq n^2. \end{aligned} \tag{5.8}$$

In the rest of this paper, we will show that under certain condition the strengthened relaxation can also help to tighten the bound up to 36%. This condition is related to the sum of squares (SOS) of tensors. In particular, a fourth order tensor $\mathcal{F} \in \Re^{n^4}$ is SOS, if there exist m symmetric matrices $A^1, \dots, A^m \in \Re^{n\times n}$ such that

$$\mathcal{F} = \sum_{j=1}^{m} A^j \otimes A^j.$$

The set of *SOS* tensors is denoted by $\Sigma^2_{n,4}$. Now we present the last result of this paper.

Proposition 5.2. *If the coefficients of objective function in problem* (5.4) *satisfy* $(c\mathcal{F} - \frac{1}{4}Q \otimes Q) \in \Sigma^2_{n,4}$, *then the strengthened model* (5.8) *can help to tighten the basic relaxation up to* $1 - \frac{2}{\pi}$.

Proof. Note that the basic relaxation (5.7)

$$\begin{aligned} \text{Maximize} \quad & P \cdot \begin{bmatrix} \mathbf{M}(\mathcal{X}) & \mathbf{V}(xx^T) \\ [1ex] \left(\mathbf{V}(xx^T)\right)^T & 1 \end{bmatrix} \\ \text{Subject to} \quad & \mathbf{M}(\mathcal{X})_{kk} = 1, \ k = 1, \dots n^2, \\ & \begin{bmatrix} \mathbf{M}(\mathcal{X}) & \mathbf{V}(xx^T) \\ [1ex] \left(\mathbf{V}(xx^T)\right)^T & 1 \end{bmatrix} \succeq 0, \end{aligned}$$

is actually in the same form of (4.1) by viewing matrix $\mathbf{M}(\mathcal{X}) \in \Re^{n^2\times n^2}$ and vector $\mathbf{V}(xx^T) \in \Re^{n^2}$ as variables. Since the coefficients of objective function in (5.4) satisfy $(c\mathcal{F} - \frac{1}{4}Q\otimes Q) \in \Sigma^2_{n,4}$, it indicates that the matrix $c\mathbf{M}(\mathcal{F}) - \frac{1}{4}\mathbf{V}(Q)\mathbf{V}(Q)^T$ is positive semidefinite. Applying Theorem 4.1, we have the result. □

Acknowledgments. The second author's research was supported by Program for Innovative Research Team of Shanghai University of Finance and Economics (IRTSHUFE) and by National Natural Science Foundation of China (NSFC) Project 11471205; the third author's research was supported in part by NSF GOALI 0800151.

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