Journal of Computational Mathematics
Vol.36, No.4, 2018, 469–491.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1701-m2016-0733

A WEAK GALERKIN FINITE ELEMENT METHOD FOR THE LINEAR ELASTICITY PROBLEM IN MIXED FORM*

Ruishu Wang and Ran Zhang
Department of Mathematics, Jilin University, Chang Chun, China
Email: wangrs16@mails.jlu.edu.cn, zhangran@mail.jlu.edu.cn

Abstract

In this paper, we use the weak Galerkin (WG) finite element method to solve the mixed form linear elasticity problem. In the mixed form, we get the discrete of proximation of the stress tensor and the displacement field. For the WG methods, we define the weak function and the weak differential operator in an optimal polynomial approximation spaces. The optimal error estimates are given and numerical results are presented to demonstrate the efficiency and the accuracy of the weak Galerkin finite element method.

Mathematics subject classification: 65N30, 65N15, 65N12, 35J50, 74B05.
Key words: Linear elasticity, mixed form, Korn's inequality, weak Galerkin finite element method.

1. Introduction

In this paper we shall present a weak Galerkin (WG) finite element method for solving the linear elasticity problem in mixed form. Let $\Omega \subset \mathbb{R}^d, (d = 2, 3)$ be an open, bounded, and connected domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$. Let Γ_D and Γ_N be two disjoint open subsets of Γ, such that $|\Gamma_D| \neq 0$, $|\Gamma_N| \neq 0$, and $\Gamma = \Gamma_D \cup \Gamma_N$. We shall consider the following problem: For a body force $\mathbf{f}$ and the initial data $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$, find a symmetric stress tensor σ and a displacement $\mathbf{u}$ such that

$$\Lambda^{-1}\sigma = \varepsilon(\mathbf{u}), \qquad \text{in } \Omega, \tag{1.1}$$

$$-\nabla\cdot\sigma = \mathbf{f}, \qquad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = \hat{\mathbf{u}}, \qquad \text{on } \Gamma_D, \tag{1.3}$$

$$\sigma\cdot\mathbf{n} = \hat{\mathbf{t}}, \qquad \text{on } \Gamma_N, \tag{1.4}$$

where Λ is assumed to be bounded, symmetric, and positive definite. For linear, homogeneous, and isotropic materials the relationship for the displacement field $\mathbf{u}$, the linear strain tensor ε, and the Cauchy stress tensor σ are given by

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t), \tag{1.5}$$

$$\sigma = \Lambda\varepsilon = 2\mu\varepsilon + \lambda\mathrm{tr}(\varepsilon)\mathrm{I}, \tag{1.6}$$

with the *Lamé* constants μ and λ defined as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)},$$

* Received August 28, 2016 / Revised version received January 16, 2017 / Accepted January 18, 2017 / Published online June 1, 2018 /

where $\nabla\mathbf{u}^t$ is the transpose for $\nabla\mathbf{u}$, E is the elasticity modulus, and ν is Poisson's ratio. In the linear elasticity problem, (1.2)-(1.4) are the balance equations with the boundary conditions. Function $\hat{\mathbf{t}}$ is the surface force density and $\mathbf{n}$ is the unit outward normal vector to the boundary.

In this paper, we shall follow the usual notations of the Sobolev spaces and the corresponding norms. For any open bounded domain $D \subset \mathbb{R}^d$ with Lipschitz continuous boundary, denote by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ the norm and the seminorm in the Sobolev space $H^s(D), s \geq 0$, respectively. When $s = 0$, the Sobolev space $H^0(D)$ coincides with the Lebesque space $L^2(D)$. Denote by $\|\cdot\|_D$ the norm in space $L^2(D)$. The inner product in spaces $L^2(D)$ and $[L^2(D)]^d$ are both denoted by $(\cdot,\cdot)_D$. The inner product on the boundary ∂D is denoted by $\langle\cdot,\cdot\rangle_{\partial D}$. If $\sigma = (\sigma_{i,j}), \tau = (\tau_{i,j}) \in [L^2(D)]^{d\times d}$, we define the inner product by

$$(\sigma,\tau)_D = \int_D \sigma : \tau d\mathbf{x}, \quad \sigma : \tau = \sum_{i,j=1}^{d} \sigma_{i,j}\tau_{i,j}.$$

For any $\mathbf{f} \in [L^2(\Omega)]^d$, $\hat{\mathbf{u}} \in [H^{\frac{1}{2}}(\Gamma_D)]^d$, and $\hat{\mathbf{t}} \in [L^2(\Gamma_N)]^d$, there exists a unique solution $(\sigma,\mathbf{u})$ to the problem (1.1)-(1.4) satisfying (see. [3])

$$\|\sigma\|_0 + \|\mathbf{u}\|_1 \leq C\Big(\|\mathbf{f}\|_{-1} + |\hat{\mathbf{u}}|_{\frac{1}{2},\Gamma_D} + |\hat{\mathbf{t}}|_{-\frac{1}{2},\Gamma_N}\Big),$$

with symmetric matrix $\sigma \in [L^2(\Omega)]^{d\times d}$ and $\mathbf{u} \in [H^1(\Omega)]^d$. Under certain conditions, for the symmetric matrix $\sigma \in [H^1(\Omega)]^{d\times d}$ and $\mathbf{u} \in [H^2(\Omega)]^d$ the following H^2-regularity holds [8,14]

$$\|\sigma\|_1 + \|\mathbf{u}\|_2 \leq C(\|\mathbf{f}\| + |\hat{\mathbf{u}}|_{\frac{3}{2},\Gamma_D} + |\hat{\mathbf{t}}|_{\frac{1}{2},\Gamma_N}),$$

where C is a positive constant independent of the Lamé constant λ. Denote by $\mathcal{S}$ the set of symmetric $d \times d$ tensors. Define

$$H(div;\mathcal{S}) = \{\tau \in [L^2(\Omega)]^{d\times d} : \ \nabla\cdot\tau \in [L^2(\Omega)]^d, \tau^t = \tau\}.$$

A weak formulation for (1.1)-(1.4) is given by: Find $\sigma \in H(div;\mathcal{S})$ and $\mathbf{u} \in [L^2(\Omega)]^d$ satisfying

$$\begin{aligned}
&(\Lambda^{-1}\sigma,\tau) + (\nabla\cdot\tau,\mathbf{u}) - \langle\tau\mathbf{n},\mathbf{u}\rangle_{\Gamma_N} = \langle\tau\mathbf{n},\hat{\mathbf{u}}\rangle_{\Gamma_D}, && \forall\tau \in H(div;\mathcal{S}),\\
&(\nabla\cdot\sigma,\mathbf{v}) - \langle\sigma\mathbf{n},\mathbf{v}\rangle_{\Gamma_N} = (-\mathbf{f},\mathbf{v}) - \langle\mathbf{v},\hat{\mathbf{t}}\rangle_{\Gamma_N}, && \forall\mathbf{v} \in [L^2(\Omega)]^d.
\end{aligned}$$

The above problem is well-posed [3].

In the mixed form, we could get the stress tensor σ directly. Otherwise, we need to drive the derivative of the displacement $\mathbf{u}$ which might lead to a loss of the accuracy. Due to the conservation law of angular momentum [13], the stress tensor is required to be symmetric. The well-posed Galerkin schemes has been explored for several decades. Important early contribution is provided in [2, 20, 21], which imposed weakly symmetry for the stress by using classical PEERS element and related approaches. A series of work is developed based on that, see [1, 4–7]. Without using the weak symmetry of the stress, [11] reports a mixed method with pseudostress-based approaches (e.g. [9, 10, 12]) for the elasticity problem. In the pseudostress-based approaches, a newly defined pseudostress variable is analyzed to show the properties of the stress tensor. In this paper, we introduce the WG method to solve the mixed form elasticity problem, which satisfies symmetry naturally.

The WG method [17,24–27] refers to a generalized finite element technique for solving partial differential equation. In the method, the differential operators are approximated by weak forms