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QUASI-NEWTON WAVEFORM RELAXATION BASED ON ENERGY METHOD*

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Abstract

A quasi-Newton waveform relaxation (WR) algorithm for semi-linear reaction-diffusion equations is presented at first in this paper. Using the idea of energy estimate, a general proof method for convergence of the continuous case and the discrete case of quasi-Newton WR is given, which appears to be the superlinear rate. The semi-linear wave equation and semi-linear coupled equations can similarly be solved by quasi-Newton WR algorithm and be proved as convergent with the energy inequalities. Finally several parallel numerical experiments are implemented to confirm the effectiveness of the above theories.

Mathematics subject classification: 65L20, 65L70.
Key words: Waveform relaxation, quasi-Newton, Energy method, Superlinear, Parallelism.

1. Introduction

Waveform relaxation has been applied to problems that model the behavior of very large-scale circuits. As an iterative solution algorithm, its superiority lies in solving large systems of time dependent equations in parallel [1, 2]. The large system of differential equations is decoupled through integrating a sequence of subsystems in fewer unknown variables within an iterative procedure. As such, waveform relaxation can be regarded as the natural extension of the classical relaxation methods with iteration vectors consisting of functions in time (waveforms) instead of scalar values [3]. The method is currently well applied for numerically solving all kinds of systems, such as differential algebraic equations (DAEs) [4,5], time-periodic problem [8], integral-differential-algebraic equations [6], fractional functional differential equations [7,9], etc. The convergence of WR is mainly discussed to present credible theory guarantee.

Then as a combination of WR and parallel-in-time algorithm parareal, parareal WR which is mainly applied for numerically solving ordinary differential equations (ODEs) is proposed, such as initial-value problem and time-periodic problem [10, 11]. Moreover, combining WR with parallel-in-space algorithm Schwarz domain decomposition generates Schwarz waveform relaxation (SWR) [12–14], which has been extensively studied recently. This time, WR method has reached high-performance scientific computing applications and usually be employed to solve partial differential equations (PDEs) indirectly.

Till the present, only a few papers have worked on directly using WR method to solve PDEs. The existing papers aiming at WR are more of limiting to ODEs that have obvious scarcities in the high number of iterations. In fact, the application prospect of WR on PDEs is of significance, such as the significant rapid way for WR method and also for PDEs [15] and immediately spreading out SWR and PWR directly on PDEs. For the proof of convergence

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of the solution with this new WR, energy estimate is a most universal method that can be implemented to a variety of PDEs. Therefore, making a connection of the advanced techniques in theory contributes to generating more useful conclusions.

On the other hand, we notice the defectiveness of classical WR when it refers to the larger scales of WR iterations and inappropriate mesh grid. By way of directly discreting the equation in time and space, the new WR algorithm aiming at semi-linear reaction-diffusion equations at the PDEs level is claimed in [15], with obvious advantages of convergence rate and the number of iteration compared with those of the classical WR method. The algorithm in [15] applies Picard relaxation to the splitting function for simplicity, whose process of iteration is relatively slow. Thus the convergence rate is expected to be improved. In this paper, we develop a theoretical framework respect to some semi-linear PDEs to study the method for its convergence behavior.

The paper is organized as follows. In Sec.2, the nonlinear term in reaction-diffusion equation is split as quasi-Newton relaxation, including two special relaxations. Then a new convergence estimate for the continuous case and the discrete case is given. Next the quasi-Newton WR algorithm is applicable to the wave equation and coupled equations with its convergence analysis in Sec.3 and Sec.4. Sec.5 provides some introduction of parallelism technique. Finally several numerical experiments directed to some kinds of equations are placed on confirming the effectiveness of the approach.

2. Quasi-Newton Waveform Relaxation Method

2.1. Reaction-diffusion equation

Let Ω be an open, bounded subset of $\mathbb{R}^n, n \geq 1$,with boundary $\partial\Omega$, and for some fixed time $T > 0$, we consider a kind of semi-linear reaction-diffusion equation for the unknown function $u(x,t) : \bar{\Omega} \times [0,T] \to \mathbb{R}$ as follow

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(u), & (\mathbf{x},t) \in \Omega \times (0,T), \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \partial\Omega \times [0,T], \\ u(\mathbf{x},0) = h(\mathbf{x}), & \mathbf{x} \in \bar{\Omega}, \end{cases} \tag{2.1}$$

where the function $h \in H_0^1(\Omega)$ and the nonlinear function $f \in C^1(\mathbb{R})$ are known.

In order to simplify the representation, we restrict ourselves to the semi-linear model in one space dimension in this paper, but the analysis that we shall present also applies in the general setting. In this regard, the iterative scheme of new WR algorithm directly at the PDEs level is

$$\begin{cases} \dfrac{\partial u^{(k+1)}}{\partial t} - \dfrac{\partial^2 u^{(k+1)}}{\partial x^2} = F(u^{(k+1)}, u^{(k)}), & (x,t) \in \Omega \times (0,T), \\ u^{(k+1)}(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u^{(k+1)}(x,0) = h(x), & x \in \bar{\Omega}. \end{cases} \tag{2.2}$$

Using the idea of quasi-linearization to nonlinear function, a parameter $\alpha \in [0,1]$ can be introduced, then the splitting function is written as

$$F(u^{(k+1)}, u^{(k)}) = (1-\alpha) f(u^{(k)}) + \alpha f(u^{(k+1)}),$$

here the linearization way for $f(u^{(k+1)})$ is replaced by Newton method

$$f(u^{(k+1)}) = f(u^{(k)}) + \frac{\partial f}{\partial u}(u^{(k)})(u^{(k+1)} - u^{(k)}),$$