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A MODIFIED PRECONDITIONER FOR PARAMETERIZED INEXACT UZAWA METHOD FOR INDEFINITE SADDLE POINT PROBLEMS*

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Abstract

The preconditioner for parameterized inexact Uzawa methods have been used to solve some indefinite saddle point problems. Firstly, we modify the preconditioner by making it more generalized, then we use theoretical analyses to show that the iteration method converges under certain conditions. Moreover, we discuss the optimal parameter and matrices based on these conditions. Finally, we propose two improved methods. Numerical experiments are provided to show the effectiveness of the modified preconditioner. All methods have fantastic convergence rates by choosing the optimal parameter and matrices.

Mathematics subject classification: 65F08, 65F10.
Key words: Preconditioner; Inexace Uzawa method; Saddle point problems; Ndefiniteness; Convergence.

1. Introduction

Consider the solution of linear equations of the 2×2 block form

$$\mathring{A}u = \begin{bmatrix} A & B^T \\ -C & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsingular, $B, C \in \mathbb{R}^{m\times n} (m < n)$ are of full row rank, $x, f \in \mathbb{R}^n$ and $y, g \in \mathbb{R}^m$, B^T denotes the transpose of the matrix B. The linear system (1.1) is called a generalized saddle point problems. Nowadays, the saddle point problems have arisen in a wide variety of engineering and scientific applications, such as computational fluid dynamics and mixed finite element approximations of elliptic PDEs and so on, see [1-3]. In addition, the solutions of the saddle point problems have also been stated and classified in detail [2,3].

As for the linear system (1.1), a large amount of works have been devoted to developing efficient iteration methods. When $\mathring{A}$ is nonsymmetric, Krylov subspace methods such as GMRES [4,5] are often applied, but such methods tend to converge slowly. In fact, it is often beneficial to employ a preconditioner in order to improve the convergence rates. And the role of preconditioners is to reduce the number of iterations required for convergence without increasing significantly the amount of computational costs required at each iteration. Thus some effective preconditioners are constructed based on matrix splitting iterative methods or matrix factorizations. They can also be constructed by special structure of the coefficient matrix [6-8]. The above methods are nonstationary iterative methods. Moreover, more stationary iterative methods are proposed when is symmetric, such as the GSOR method [9], the HSS-like methods

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[10-11], the parameterized inexact Uzawa methods [12-14], and some new combined methods like the Uzawa-SOR method [15] and the Uzawa-HSS method [16]. These stationary methods require much less computer memory than nonstationary methods in actual implementation. However, they may be less efficient in some situations. The iterative methods become more attractive than the direct methods for solving the saddle point problems, but direct methods play an important role in the form of preconditioners embedded in an iterative framework even for some stationary methods.

In this paper, we modify the preconditioner to improve the performance and solve more general situations for the indefinite saddle point problems in (1.1) by giving a new variable parameter. At first, the preconditioning matrix is applied, and we get the iteration matrix by splitting the coefficient matrix, and give the corresponding convergence analysis in brief. Then, the optimal parameter and matrices are discussed in detail. Finally, we propose and analyze several methods. The remainder of the paper is organized as follows. In section 2, we modify the existing preconditioner [20] for the parameterized inexact Uzawa methods and analyze its convergence. In section 3, we discuss the parameter and matrices by considering Theorem 2.1 presented in section 2, and give the optimal choice. Moreover, two methods are derived by different choices of the parameter and matrices in section 4 under the assumption of Theorem 3.1. In section 5, we use a numerical example to show the fast convergence of the methods proposed, which also shows that the modified method are efficient and powerful.

2. Modified Preconditioned Parameterized Inexact Uzawa Methods

First of all, we give a modified matrix P which is nonsingular for preconditioning the linear system (1.1):

$$P = \begin{bmatrix} R_1 A^{-1} & O \\ B(A^{-1})^T C^T R_2 C A^{-1} & B(A^{-1})^T C^T R_2, \end{bmatrix}$$

where $R_1 \in \mathbb{R}^{n\times n}$ and $R_2 \in \mathbb{R}^{m\times m}$ are symmetric positive definite. Then we get the new linear system of the form

$$\mathring{A} u = b, \tag{2.1}$$

where

$$\text{Ł} = P\mathring{A} = \begin{bmatrix} R_1 & R_1 A^{-1} B^T \\ O & B(A^{-1})^T C^T R_2 C A^{-1} B^T \end{bmatrix},$$

$$d = Pb = \begin{bmatrix} R_1 A^{-1} f \\ B(A^{-1})^T C^T R_2 (C A^{-1} f + g) \end{bmatrix}.$$

Note that

$$B(A^{-1})^T C^T R_2 C A^{-1} B^T = (C A^{-1} B^T)^T R_2 (C A^{-1} B^T),$$

R_2 is symmetric positive definite, B, C are of full row rank and A is nonsingular. Then $B(A^{-1})^T C^T R_2 C A^{-1} B^T$ is symmetric positive definite. Because R_1 is also symmetric positive definite, it is easily found that Ł is nonsingular. Then we consider the following matrix splitting:

$$\text{Ł} = \begin{bmatrix} R_1 + Q_1 & O \\ Q_3 & Q_2 \end{bmatrix} - \begin{bmatrix} Q_1 & -R_1 A^{-1} B^T \\ Q_3 & Q_2 - B(A^{-1})^T C^T R_2 C A^{-1} B^T \end{bmatrix},$$

where $R_1 + Q_1$ and Q_2 are nonsingular, and $Q_3 \in \mathbb{R}^{m\times n}$ is arbitrary.