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HETEROGENEOUS MULTISCALE METHOD FOR OPTIMAL CONTROL PROBLEM GOVERNED BY ELLIPTIC EQUATIONS WITH HIGHLY OSCILLATORY COEFFICIENTS*

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Abstract

In this paper, we investigate heterogeneous multiscale method (HMM) for the optimal control problem with distributed control constraints governed by elliptic equations with highly oscillatory coefficients. The state variable and co-state variable are approximated by the multiscale discretization scheme that relies on coupled macro and micro finite elements, whereas the control variable is discretized by the piecewise constant. By applying the well-known Lions' Lemma to the discretized optimal control problem, we obtain the necessary and sufficient optimality conditions. A priori error estimates in both L^2 and H^1 norms are derived for the state, co-state and the control variable with uniform bound constants. Finally, numerical examples are presented to illustrate our theoretical results.

Mathematics subject classification: 65N06, 65B99.
Key words: Constrained convex optimal control, Heterogeneous multiscale finite element, A priori error estimate, Elliptic equations with highly oscillatory coefficients.

1. Introduction

In modern scientific and engineering computation, many important practical problems are complicated and multiscale ones, such as composite materials with thermal/electrical conductivity, flow through the heterogeneous porous media, and time scale of the chemical reactions, etc. These multiscale problems are often described by PDEs with highly oscillatory coefficients [20]. Many kinds of multiscale computation methods have been researched to deal with

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the intrinsic complexity of the multiscale problems, such as the multigrid method [5,35], adaptive method [11], domain decomposition method [34], homogenization method [4], wavelet-based numerical homogenization method [10], partition of unity method [32], multiscale finite element method [17], heterogeneous multiscale method [12], variational multiscale method [16], etc.

Also, the optimal control problems governed by PDEs with highly oscillatory coefficients arise in many real-world problems, including the above-mentioned problems for composite materials and porous media. Due to the intrinsic complexity of the state equations, it is very difficult to derive the direct numerical solution for these problems. An important numerical method is the homogenization method, which could give the overall behavior by incorporating the fluctuations due to the heterogeneities [25]. There have been many works considering this method, such as [14, 19, 23, 37]. Another important numerical method is the multiscale asymptotic method. Lions [24] presented the asymptotic expansions for the optimal control with small parameter ε. Cao considered the multiscale asymptotic expansions to the boundary control [6], optimal control for elliptic systems with constraints [25] and optimal control for linear parabolic equations [7]. However, there exists less work about multiscale finite element method for optimal control problems. To our best knowledge, we only find some results for the mulitscale finite element method for optimal control problems without constraint in [20], which have a very different nature compared with the constrained control problems.

In this article, we apply the heterogeneous multiscale finite element method [1–3, 12, 13] to solve the optimal control problem governed by elliptic equations with highly oscillatory coefficients. We derive the continuous and discrete first-order optimality conditions by the Lagrange multiplier method. And we prove the a priori error estimates in L^2 and H^1 norms for the state variable, the co-state variable, and the control variable with uniform bound constants. Numerical examples are given to illustrate the validity of the estimates. Compared with other multiscale finite element methods for optimal control problem such as [20], our work achieves improved results in the convergence order. Although the result given in [25] was also of ε, that work used the multiscale asymptotic analysis.

Our article is organized as follows. In the next section, we present the mathematical setting and optimality conditions for the elliptic optimal control problems with highly oscillatory coefficients. In Section 3, the heterogeneous multiscale finite element scheme for the optimal control problem is given and the discretized optimality conditions are obtained. In Section 4, we prove a priori error estimates in L^2 and H^1 norms for the state variable and the co-state variable, and in L^2 norm for the control variable, respectively. In Section 5, we present the numerical examples to confirm our theoretical findings. Finally, we draw some concluding remarks in Section 6.

In what follows, $C > 0$ or $c > 0$ denotes a generic constant, independent of ε, h and h_U. We denote $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and $\Omega_U \subset \mathbb{R}^d$ is another bounded domain with Lipschitz boundary $\partial\Omega_U$. Generally, Ω_U can be a subset of Ω. In the special case, we take $\Omega_U = \Omega$. We adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev space on Ω with a norm $\|\cdot\|_{m,q,\Omega}$ and a seminorm $|\cdot|_{m,q,\Omega}$. Let

$$W_0^{m,q}(\Omega) = \Big\{ v \in W^{m,q}(\Omega) : v|_{\partial\Omega} = 0 \Big\}, \quad H^m(\Omega) = W^{m,2}(\Omega), \quad \|\cdot\|_{m,\Omega} = \|\cdot\|_{m,2,\Omega}.$$

Especially, we take the state space is $H_0^1(\Omega)$, and the control space is $L^2(\Omega_U)$. The inner products in $L^2(\Omega_U)$ and $L^2(\Omega)$ are indicated by $(\cdot,\cdot)_U$ and $(\cdot,\cdot)$, respectively.