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OPTIMAL QUADRATIC NITSCHE EXTENDED FINITE ELEMENT METHOD FOR INTERFACE PROBLEM OF DIFFUSION EQUATION*

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Abstract

In this paper, we study Nitsche extended finite element method (XFEM) for the interface problem of a two dimensional diffusion equation. Specifically, we study the quadratic XFEM scheme on some shape-regular family of grids and prove the optimal convergence rate of the scheme with respect to the mesh size. Main efforts are devoted onto classifying the cases of intersection between the elements and the interface and prove a weighted trace inequality for the extended finite element functions needed, and the general framework of analysing XFEM can be implemented then.

Mathematics subject classification: 65N30, 65N12, 65N15.
Key words: Interface problems, Extended finite element methods, Error estimates, Nitsche's scheme, Quadratic element.

1. Introduction

Many problems in physics, engineering, and other fields contain a certain level of coupling between different physical systems, such as the coupling between fluid and structure in fluid-structure interaction problems, and the coupling among different flows in multi-phase flows problems. An interface where the coupling takes place is generally encountered in such kind of problems, and, consequently, the numerical discretization of the interface problem is important in applied sciences and mathematics. In this paper, we take the diffusion equation

$$-\nabla \cdot (\alpha(x)\nabla u) = f, \tag{1.1}$$

as a model problem, and study its interface problem. Namely, the underlying domain is assumed to be divided to two subdomains by an interface, such that α is smooth on each subdomain, but not smooth on the whole domain. The model problem is a fundamental one in numerical analysis, and very incomplete literature review can be found in, e.g., [4, 5, 14, 30, 31] and below.

Different from problem with smooth coefficient, the existence of an interface for diffusion equation can invalidate the global smoothness of the solution of the system, and the accuracy of the standard finite element method is limited when used for such problems. As the loss of

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accuracy occurs near the interface, where the solution is no longer smooth, the obstacle can be circumvented by implementing a body-fitted (or interface-resolved) grid. In this approach, the "bad part" of the solution that may not be well approximated is restricted in a narrow region surrounding the interface. We refer the reader to [30] for an asymptotic optimal error estimate on body-fitted grid:

$$\|u-u_I\|_{0,\Omega}+h|u-u_I|_{1,\Omega}\le C|\log h|^{1/2}h^2|u|_{2,\Omega_1\cup\Omega_2},$$

where u_I is the linear element interpolation for $u\in H^1(\Omega)\cap H^2(\Omega_1\cup\Omega_2)$. We use $|w|_{m,\Omega_1\cup\Omega_2}$ or $\|w\|_{m,\Omega_1\cup\Omega_2}$ to denote $|w|_{m,\Omega_1}+|w|_{m,\Omega_2}$ or $\|w\|_{m,\Omega_1}+\|w\|_{m,\Omega_2}$, respectively, for $w\in H^m(\Omega_1\cup\Omega_2):=\{v\in L^2(\Omega):v|_{\Omega_i}\in H^m(\Omega_i),\ i=1,2\}$. A sharper estimate without the logarithmic factor was given in Bramble and King [4] (1996):

$$\|u-u_I\|_{0,\Omega}+h|u-u_I|_{1,\Omega}\leqslant Ch^2|u|_{2,\Omega_1\cup\Omega_2}. \tag{1.2}$$

We also refer to [5, 14, 31] for related discussions. Unfortunately, it is usually a nontrivial and time-consuming task to construct good interface-fitted meshes for problems involving geometrically complicated interfaces. Another idea is to add some basis functions which can be non-smooth around the interface so that the nonsmooth part of the solution can be accurately captured. By enriching the finite element space with local basis functions that can resolve the interface, numerous modified methods based only on simple Cartesian grids are proposed. For finite element methods, we refer to the work of [6, 16] for elliptic problems with discontinuous coefficients, where finite element basis functions are. modified at the coefficients discontinuity. Extra work around the interface can also be recognised implemented in the finite difference setting; we refer to, e.g., [24] for the immersed boundary method, to [13, 15] for the immersed interface method, to [19] for the ghost fluid method and etc..

The extended finite element method (XFEM) falls into the category of the second approach. The XFEM extends the classical finite element method by enriching the solution space locally around the interface with nonsmooth functions, and it is realized through the partition of unity concept. Developed by Belytschko and his collaborators ([3,21]), the XFEM is originally used for modelling crack growth, and has now been used widely in crack propagation ([3,8,21,22]), fluid-structure interaction (FSI) ([9, 28]), multi-phase flows ([10]), and other multi-physics problems that involve interfaces. This approach provides accurate approximation for the problems with jumps, singularities, and other locally nonsmooth features within elements. We refer to [7] and the reference therein for a historical account for XFEM.

The Nitsche-XFEM scheme presented in [11] is a special kind of XFEM which can also be regarded as the coupling of the continuous finite element method and discontinuous Galerkin (DG) method. In [11], Nitsche's formulation of an elliptic interface problem was introduced, and then the extended finite element space is constructed by enriching the standard finite element space with functions that are completely discontinuous across the interface; stabilisation terms are then needed for the stability of the scheme. The Nitsche-XFEM uses essentially piecewise polynomials and flexibility can be expected respectively in designing schemes of various orders and in implementing. The optimality of the Nitsche-XFEM scheme for interface problem is proved when the XFE space consists of linear continuous piecewise basis functions and extended basis functions constructed by partition of unity in [11]. Technically, a crucial fact is that the gradient of a linear polynomial is constant. This fact does not hold for polynomials of higher degrees, and so far as we know, the analysis of the high-order Nitsche-XFEM is still absent.