Journal of Computational Mathematics
Vol.36, No.6, 2018, 792–832.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1705-m2016-0545

FINITE ELEMENT EXTERIOR CALCULUS FOR PARABOLIC EVOLUTION PROBLEMS ON RIEMANNIAN HYPERSURFACES*

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Abstract

Over the last ten years, Finite Element Exterior Calculus (FEEC) has been developed as a general framework for linear mixed variational problems, their numerical approximation by mixed methods, and their error analysis. The basic approach in FEEC, pioneered by Arnold, Falk, and Winther in two seminal articles in 2006 and 2010, interprets these problems in the setting of *Hilbert complexes*, leading to a more general and complete understanding. Over the last five years, the FEEC framework has been extended to a broader set of problems. One such extension, due to Holst and Stern in 2012, was to problems with *variational crimes*, allowing for the analysis and numerical approximation of linear and geometric elliptic partial differential equations on Riemannian manifolds of arbitrary spatial dimension. Their results substantially generalize the existing surface finite element approximation theory in several respects. In 2014, Gillette, Holst, and Zhu extended FEEC in another direction, namely to parabolic and hyperbolic evolution systems by combining the FEEC framework for elliptic operators with classical approaches for parabolic and hyperbolic operators, by viewing solutions to the evolution problem as lying in Bochner spaces (spaces of Banach-space valued parametrized curves). Related work on developing an FEEC theory for parabolic evolution problems has also been done independently by Arnold and Chen. In this article, we extend the work of Gillette-Holst-Zhu and Arnold-Chen to evolution problems on Riemannian manifolds, through the use of framework developed by Holst and Stern for analyzing variational crimes. We establish *a priori* error estimates that reduce to the results from earlier work in the flat (non-criminal) setting. Some numerical examples are also presented.

Mathematics subject classification: 65M15, 65M60, 53C44
Key words: FEEC, Elliptic equations, Evolution equations, Approximation theory, Inf-sup conditions, *A priori* estimates, Variational crimes, Equations on manifolds

1. Introduction

Arnold, Falk, and Winther [2, 3] introduced Finite Element Exterior Calculus (FEEC) as a general framework for linear mixed variational problems, their numerical approximation by mixed methods, and their error analysis. They recast these problems using the ideas and tools of *Hilbert complexes*, leading to a more complete understanding. Subsequently, Holst and Stern [24] extended the Arnold–Falk–Winther framework to include *variational crimes*, allowing for the analysis and numerical approximation of linear and geometric elliptic partial differential equations on Riemannian manifolds of arbitrary spatial dimension, generalizing the

* Received February 18, 2016 / Revised version received October 7, 2016 / Accepted May 8, 2017 / Published online August 7, 2018 /

existing surface finite element approximation theory in several directions. Gillette, Holst, and Zhu [22] extended FEEC in another direction, namely to parabolic and hyperbolic evolution systems by combining recent work on FEEC for elliptic problems with a classical approach of Thomée [35] to solving evolution problems using semi-discrete finite element methods, by viewing solutions to the evolution problem as lying in Bochner spaces (spaces of Banach-space valued parametrized curves). Arnold and Chen [1] independently developed related work, for generalized Hodge Laplacian parabolic problems for differential forms of arbitrary degree, and Holst, Mihalik, and Szypowski [23] consider similar work in adaptive finite element methods.

In this article, we aim to combine the approaches of the above articles, extending the work of Gillette, Holst, and Zhu [22] and Arnold and Chen [1] to evolution problems in abstract Hilbert complexes by using the framework of Holst and Stern [24]. We then apply the results to parabolic problems on Riemannian hypersurfaces approximated by piecewise polynomial curved triangulations in a tubular neighborhood, using piecewise polynomial finite element spaces. As in earlier literature on finite elements on approximating surfaces by Dziuk [17], Dziuk and Demlow [16], and Demlow [15], the error splits into a PDE approximation term and a surface approximation term. An interesting result that follows is that the optimal rate of convergence occurs when the polynomial degree of both the approximating surfaces and the finite element spaces are the same (the isoparametric case). Similar observations have been made for the surface finite element method [15] and in the isogeometric analysis literature [12, 26].

1.1. The Hodge heat equation and its mixed form

We now motivate our problem with a concrete example. We consider an evolution equation for differential forms on a manifold, and then we rephrase it as a mixed problem as an intermediate step toward semidiscretization using mixed finite element methods. We then see how this allows us to leverage existing *a priori* error estimates for parabolic problems, and see how it fits in the framework of Hilbert complexes.

Let M be a smooth compact oriented Riemannian n-manifold without boundary embedded in $\mathbb{R}^{n+1}$. The **Hodge heat equation** is to find time-dependent k-form

$$u : M \times [0, T] \to \Lambda^k(M)$$

(where $\Lambda^k(M)$ denotes the bundle of alternating k-tensors on M) such that

$$\begin{aligned} &u_t - \Delta u = u_t + (\delta d + d\delta)u = f \quad \text{in } M, \quad \text{for } t > 0 \\ &u(\cdot, 0) = g \qquad\qquad\qquad\qquad\qquad \text{in } M. \end{aligned} \tag{1.1}$$

where g is an initial k-form, and f, a possibly time-dependent k-form, is a source term. Note that no boundary conditions are needed since $\partial M = \emptyset$. This is the problem studied by Arnold and Chen [1], and in the case $k = n$, one of the problems studied by Gillette, Holst, and Zhu [22], building upon work in special cases for domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ by Johnson and Thomée [27, 35]

For the stability of the numerical approximations with the methods of [25] and [3], we recast the problem in mixed form, converting the problem into a system of differential equations. Motivating the problem by setting $\sigma = \delta u$ (recall that for the Dirichlet problem and $k = n$, δ here corresponds to the *negative* of the gradient in Euclidean space, and is the adjoint of d, corresponding to the divergence), and taking the adjoint, we have

$$\begin{aligned} &\langle \sigma, \omega \rangle - \langle u, d\omega \rangle = 0, && \forall\, \omega \in H\Omega^{k-1}(M), \quad t > 0, \\ &\langle u_t, \varphi \rangle + \langle d\sigma, \varphi \rangle + \langle du, d\varphi \rangle = \langle f, \varphi \rangle, && \forall\, \varphi \in H\Omega^k(M) \quad t > 0, \\ &u(0) = g \end{aligned} \tag{1.2}$$

where $H\Omega^k(M)$ denotes the Sobolev space of k-forms for which weak exterior derivatives exist. We do not have to explicitly account for harmonic forms in the formulation of the parabolic equations, because they will evolve in time with the rest of the solution. This contrasts to the case of elliptic equations, where the goal is to invert an operator; harmonic forms directly obstruct this process. Harmonic forms will, however, definitely play a critical role in our analysis and bring new results not apparent in the $k = n$ case.

1.2. Semidiscretization of the equation

In order to analyze the numerical approximation, we *semidiscretize* our problem; that is, we discretize in space and leave time continuous. We first note that M can be regarded as the zero level set of some signed distance function, and therefore has some tubular neighborhood U on which the distance function is smooth up to some absolute distance $\delta_0 > 0$ (a standard result in Riemannian geometry). To discretize, following Holst and Stern [24], we consider a collection of approximating surfaces M_h to M, all contained in U, and a projection $a : M_h \to M$ along the surface normal of M. They assume the approximating surfaces are constructed (in a manner we shall describe shortly) from shape-regular and quasi-uniform triangulations $\mathcal{T}_h$ (writing T_h for the corresponding piecewise linear manifold $\bigcup \mathcal{T}_h$), where h is the maximum diameter of the elements. The piecewise linear case $M_h = T_h$ was studied initially by Dziuk [17] and later by Demlow and Dziuk [16], and the piecewise polynomial case, obtained by Lagrange interpolation of the projection a over T_h, was later studied by Demlow [15]. We pull forms on M_h to M back via the inverse of the normal projection, which furnishes injective morphisms $i_h^k : \Lambda_h^k \hookrightarrow H\Omega^k(M)$ as required by the theory in [24], which we shall review in Section 2 below. Finally, we need a family of linear projections $\Pi_h^k : H\Omega^k(M) \to \Lambda_h^k$ such that $\Pi_h \circ i_h = \mathrm{id}$ which allow us to interpolate given data into the chosen finite element spaces—this is necessary because some of the more obvious, natural seeming choices of operators, such as the adjoint i_h^* to i_h, can be difficult to compute (i_h^* will still be useful in our analysis).

We now formulate the semidiscrete problem: we seek a solution $(\sigma_h, u_h) \in H_h \times S_h \subseteq H\Omega^{k-1} \times H\Omega^k$ such that

$$\begin{aligned}
&\langle \sigma_h, \omega_h \rangle_h - \langle u_h, d\omega_h \rangle_h = 0, && \forall\, \omega_h \in H_h, \quad t > 0 \\
&\langle u_{h,t}, \varphi_h \rangle_h + \langle d\sigma_h, \varphi_h \rangle_h + \langle du_h, d\varphi_h \rangle_h = \langle \Pi_h f, \varphi_h \rangle_h, && \forall\, \varphi_h \in S_h \quad t > 0 \\
&u_h(0) = g_h.
\end{aligned} \tag{1.3}$$

We shall describe how to define $g_h \in S_h$ shortly; it is to be some suitable interpolation of g. As S_h and H_h are finite-dimensional spaces, we can reduce this to a system of ODEs in Euclidean space by choosing bases (ψ_i) for H_h and (ϕ_k) for S_h; expanding the unknowns $\sigma_h = \sum_i \Sigma^i(t)\psi_i$ and $u_h = \sum_k U^k(t)\phi_k$; substituting these basis functions as test functions to form matrices $A_{k\ell} = \langle \phi_k, \phi_\ell \rangle$, $B_{ik} = \langle d\psi_i, \phi_k \rangle$, $K_{k\ell} = \langle d\phi_k, d\phi_\ell \rangle$, and $D_{ij} = \langle \psi_i, \psi_j \rangle$; forming the vectors for the load data F defined by $F_k = \langle F, \phi_k \rangle$; and finally forming the initial condition vector G from $g_h = \sum G^k \phi_k$. We thus arrive at the matrix equations for the unknown, time-dependent coefficient vectors Σ and U:

$$\begin{aligned}
&D\Sigma - B^T U = 0, \\
&AU_t + B\Sigma + KU = F, \qquad \text{for } t > 0 \\
&U(0) = G.
\end{aligned}$$

The matrices A and D are positive-definite, hence invertible. Substituting $\Sigma = D^{-1}B^TU$, we have the system of ODEs

$$AU_t + (BD^{-1}B^T + K)U = F, \text{ for } t > 0, \quad U(0) = G,$$

which has a unique solution by standard ODE theory. For purposes of actually numerically integrating the ODE, namely, discretizing fully in space and time, it is better not to use the above formulation, because it can lead to dense matrices. Computationally, this is due to the explicit presence of an inverse, D^{-1}, not directly multiplying the variable; conceptually, this is actually a statement about the discrete adjoint to the codifferential d_h^* generally having global support even if the finite element functions are only locally supported [1]. Instead, we differentiate the first equation with respect to time, getting $D\Sigma_t - B^TU_t = 0$, which leads to the block system

$$\frac{d}{dt}\begin{pmatrix} D & -B^T \\ 0 & A \end{pmatrix}\begin{pmatrix} \Sigma \\ U \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -B & -K \end{pmatrix}\begin{pmatrix} \Sigma \\ U \end{pmatrix} + \begin{pmatrix} 0 \\ F \end{pmatrix} \tag{1.4}$$

which is still well-defined ODE for Σ and U, as the invertible matrices A and D appear on the diagonal. The continuous analogue of this differentiated equation also plays a role in the showing that the continuous problem is well-posed.

These equations differ from those studied by Gillette, Holst, and Zhu [22], Arnold and Chen [1], and Thomée [35] by the choice of finite element spaces—here we are assuming them to be differential forms with piecewise polynomial coefficients defined on a triangulated mesh in a tubular neighborhood of a manifold, rather than being defined on subsets of Euclidean space. This suggests that we should try to gather these commonalities, examine what happens in abstract Hilbert complexes, and see how general a form of error estimate we can get this way.

1.3. Error analysis

The general idea of the method of Thomée [35] is to compare the semidiscrete solution to an **elliptic projection** of the data, a method first explored by Wheeler [37]. If we assume that we already have a solution u to the continuous problem, then for each fixed time t, $u(t)$ can be considered as trivially solving an elliptic equation with data $-\Delta u(t)$. Thus, using the methods developed in [3], we consider the discrete solution $\tilde{u}_h$ for u in this elliptic problem (namely, applying the discrete solution operator T_h to $-\Delta u(t)$). This may be compared to the true solution (at each fixed time) using the error estimates in [3]. What remains is to compare the semidiscrete solution u_h (as defined by the ODEs (1.3) above) to the elliptic projection, so that we have the full error estimate by the triangle inequality. Thomée derives the following estimates, for finite elements in the plane ($n = 2$) of top-degree forms ($k = 2$, there represented by a scalar proxy), for g_h the elliptic projection of the initial condition g and $t \geq 0$:

$$\|u_h(t) - u(t)\|_{L^2} \leq ch^2 \left(\|u(t)\|_{H^2} + \int_0^t \|u_t(s)\|_{H^2} ds \right), \tag{1.5}$$

$$\|\sigma_h(t) - \sigma(t)\|_{L^2} \leq ch^2 \left(\|u(t)\|_{H^3} + \left(\int_0^t \|u_t(s)\|_{H^2}^2 ds \right)^{1/2} \right). \tag{1.6}$$

Gillette, Holst, and Zhu [22], and Arnold and Chen [1] generalize these estimates and represent them in terms of Bochner norms. These estimates describe the accumulation of error up to

fixed time value t, assuming, of course, that all the functions involved are sufficiently regular to allow those estimates. The key equation that makes these estimates possible are Thomée's error evolution equations: defining $\rho = \|\tilde{u}_h(t) - u(t)\|$, $\theta = \|u_h(t) - \tilde{u}_h(t)\|$, and $\varepsilon = \|\sigma_h(t) - \tilde{\sigma}_h(t)\|$, we have

$$\begin{aligned}
&\langle \theta_t, \phi_h \rangle - \langle \operatorname{div} \varepsilon(t), \phi_h \rangle = -\langle \rho_t, \phi_h \rangle, \\
&\langle \varepsilon, \omega_h \rangle + \langle \theta, \operatorname{div} \omega_h \rangle = 0.
\end{aligned}$$

These are used to derive certain differential inequalities and make Grönwall-type estimates. In this chapter, we examine the above error equations and place them in a more abstract framework. We use Bochner spaces (also used by [22]) to describe time evolution in Hilbert complexes, building on their successful use in elliptic problems. We investigate Thomée's method in this framework to gain further insight into how finite element error estimates evolve in time.

1.4. Summary of the paper

The remainder of this paper is structured as follows. In Section 2, we review the finite element exterior calculus (FEEC) and the variational crimes framework of Holst and Stern [24]. We prove some extensions in order to account for problems with prescribed harmonic forms; this is what allows the elliptic projection to work in the case where harmonic forms are present. In Section 3, we formulate abstract parabolic problems in Bochner spaces and extend some standard results on the existence and uniqueness of strong solutions, and describe how this problem fits into that framework. In Section 4, we extend the *a priori* error estimates for Galerkin mixed finite element methods to parabolic problems on Hilbert complexes. Then, we relate the resuls to the problem on manifolds. The main abstract result is Theorem 4.1, which uses the previous results from the FEEC framework with variational crimes, in order to understand how those error terms evolve with time. We then specialize, in Section 5 to parabolic equations on Riemannian manifolds, the motivating example, and see how this generalizes the error estimates of Thomée [35], Gillette, Holst, and Zhu [22], and Holst and Stern [24]. In Section 6, we give some remarks on how to implement the method.

2. Finite Element Exterior Calculus

We review here the relevant results from the finite element exterior calculus (FEEC) that we will need for this paper. FEEC was introduced in Arnold, Falk and Winther [2, 3] as a framework for deriving error estimates and formulating stable numerical methods for a large class of elliptic PDE. One of the central ideas which helped unify many of these distinct methods into a structured framework has been the idea of **Hilbert complexes** [8], which abstracts the essential features of the cochain complexes commonly found in exterior calculus and places them in a context where modern methods of functional analysis may be applied. This assists in formulating and solving boundary value problems, in direct analogy to how Sobolev spaces have helped provide a framework for solving such problems for scalar functions. Arnold, Falk, and Winther [3] place numerical methods into this framework by choosing certain finite-dimensional subspaces satisfying certain compatibility and approximation properties. Holst and Stern [24] extended this framework by considering the case in which there is an injective morphism from a finite-dimensional complex to the complex of interest, without it necessarily being inclusion. This allows the treatment of geometric **variational crimes** [6, 7], where an approximating

manifold (on which it may be far easier to choose finite element spaces) no longer coincides with the actual manifold on which we seek our solution. We review the theory as detailed in [24] and refer the reader there for details.

2.1. Hilbert Complexes

As stated before, the essential details of differential complexes, such as the de Rham complex, are nicely captured in the notion of Hilbert complexes. This enables us to see clearly where many elements of boundary value problems come from, in particular, the Laplacian, Hodge decomposition theorem, and Poincaré inequality. In addition, it allows us to see how to carry these notions over to numerical approximations.

Definition 2.1 (Hilbert complexes) *We define a **Hilbert complex** (W, d) to be sequence of Hilbert spaces W^k with possibly unbounded linear maps $d^k : V^k \subseteq W^k \to V^{k+1} \subseteq W^{k+1}$, such that each d^k has closed graph, is densely defined, and satisfies the **cochain property** $d^k \circ d^{k-1} = 0$ (this is often abbreviated $d^2 = 0$; we often omit the superscripts when the context is clear). We call each V^k the **domain** of d^k. We will often refer to elements of such Hilbert spaces as "forms," being motivated by the canonical example of the de Rham complex. If (W, d) is a Hilbert complex, and $S^k \subseteq W^k$ is a sequence of subspaces such that (S, d) is also a Hilbert complex, that is, each d^k on W^k restricts to $S^k \cap V^k$ into $S^{k+1} \cap V^{k+1}$, (S, d) is called a **subcomplex** of (W, d). The Hilbert complex is called a **closed complex** if each image space $\mathfrak{B}^k = d^{k-1}V^{k-1}$ (called the **k-coboundaries**) is closed in W^k, and a **bounded complex** if each d^k is in fact a bounded linear map. The most common arrangement in which one finds a bounded complex is by taking the sequence of domains V^k, the same maps d^k, but now with the **graph inner product***

$$\langle v, w\rangle_V = \langle v, w\rangle + \langle d^k v, d^k w\rangle.$$

*for all $v, w \in V^k$; this complex (V, d) is called the **domain complex**. Unsubscripted inner products and norms will always be assumed to be the ones associated to W^k, and for emphasis, these relevant quantities will be prefixed with W- or V- (such as "W-orthogonal", "V-inner product", etc.) as necessary.*

Definition 2.2 (Cocycles, Coboundaries, and Cohomology). *The kernel of the map d^k in V^k will be called $\mathfrak{Z}^k$, the **k-cocycles** and, as before, we have $\mathfrak{B}^k = d^{k-1}V^{k-1}$. Since $d^k \circ d^{k-1} = 0$, we have $\mathfrak{B}^k \subseteq \mathfrak{Z}^k$, so we may form the quotient spaces $\mathfrak{Z}^k/\mathfrak{B}^k$, called the **$k$th cohomology group**.*

Definition 2.3 (Dual complexes and adjoints). *For a Hilbert complex (W, d), we can form the **dual complex** (W^*, d^*) which consists of spaces $W_k^* = W^k$, maps $d_k^* : V_k^* \subseteq W_k^* \to V_{k-1}^* \subseteq W_{k-1}^*$ such that $d_{k+1}^* = (d^k)^*$, the adjoint operator satisfying*

$$\langle d_{k+1}^* v, w\rangle = \langle v, d^k w\rangle.$$

The operators d^* decrease degree, so this is a chain complex, rather than a cochain complex; the analogous concepts to cocycles and coboundaries extend to this case and we write $\mathfrak{Z}_k^*$ and $\mathfrak{B}_k^*$ for them.

Definition 2.4 (Morphisms of Hilbert complexes). *Let (W, d) and (W', d') be two Hilbert complexes with domain complexes (V, d) and (V', d), respectively. $f : W \to W'$ is called a **morphism of Hilbert complexes** if we have a sequence of bounded linear maps $f^k : W^k \to W'^k$ such that $f(V^k) \subseteq V'^k$ and $d'^k \circ f^k = f^{k+1} \circ d^k$ (i.e., they **commute** with the differentials).*

We note that for a subcomplex (S, d) of (W, d), the inclusion maps $i^k : S^k \hookrightarrow W^k$ are morphisms of Hilbert complexes.

With the above, we can show the following **w**eak Hodge decomposition:

Theorem 2.1 (Hodge Decomposition Theorem). *Let (W, d) be a Hilbert complex with domain complex (V, d). The **harmonic space** $\mathfrak{H}^k$ is the W-orthogonal complement of $\mathfrak{B}^k$ in $\mathfrak{Z}^k$. This means, by the orthogonal decomposition theorem in Hilbert spaces, that $\mathfrak{Z}^k = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k$. Then we have the W- and V-orthogonal decompositions*

$$W^k = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W} \tag{2.1}$$

$$V^k = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V}. \tag{2.2}$$

where $\mathfrak{Z}^{k\perp_V} = \mathfrak{Z}^{k\perp_W} \cap V^k$.

Of course, if $\mathfrak{B}^k$ is closed, then the extra closure is unnecessary, and we omit the term "weak". We shall simply write $\mathfrak{Z}^{k\perp}$ for $\mathfrak{Z}^{k\perp_V}$, which will be the most useful orthogonal complement for our purposes. The orthogonal projections P_U for a subspace U will be in the W-inner product unless otherwise stated. We note that by the abstract properties of adjoints ([3, §3.1.2]), $\mathfrak{Z}^{k\perp_W} = \overline{\mathfrak{B}_k^*}$, and $\mathfrak{B}^{k\perp_W} = \mathfrak{Z}_k^*$. Also very useful is that the V- and W-inner products (and therefore, norms and orthogonal projections) agree on $\mathfrak{Z}$ and hence on $\mathfrak{B}$ and $\mathfrak{H}$.

The following inequality is an important result crucial to the stability of our solutions to the mixed abstract Hodge Laplacian problem (the next definition), as well as its numerical approximation:

Theorem 2.2 (Abstract Poincaré Inequality). *If (V, d) is a closed, bounded Hilbert complex, then there exists a constant $c_P > 0$ such that for all $v \in \mathfrak{Z}^{k\perp}$,*

$$\|v\|_V \le c_P \|d^k v\|_V.$$

In the case that (V, d) is the domain complex associated to a closed Hilbert complex (W, d), (V, d) is again closed, and the additional graph inner product term vanishes: $\|d^k v\|_V = \|d^k v\|$. We now introduce the abstract version of the Hodge Laplacian and the associated problem.

Definition 2.5 (Abstract Hodge Laplacian problems). *Consider $\Delta := -(dd^* + d^*d)$ on a Hilbert complex (W, d), called the **abstract Hodge Laplacian**.*[1)] *Its domain is $D_L = \{u \in V^k \cap V_k^* : du \in V_{k+1}^*, d^*u \in V^{k-1}\}$, and the **Hodge Laplacian problem** is to seek $u \in V^k \cap V_k$, given $f \in W^k$, such that*

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle \tag{2.3}$$

*for all $v \in V^k \cap V_k^*$. This is simply the weak form of the Laplacian and any $u \in V^k \cap V_k^*$ satisfying the above is called a **weak solution**. Owing to difficulties in the approximation theory for such a problem (it is difficult to construct finite elements for the space $V^k \cap V_k^*$), Arnold, Falk, and Winther [3] formulate the **mixed abstract Hodge Laplacian problem** by defining auxiliary variables $\sigma = d^*u$ and $p = P_{\mathfrak{H}} f$, the orthogonal projection of f into the harmonic space, and consider a* system *of equations, to seek $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ such that*

$$\begin{aligned} &\langle \sigma, \tau\rangle - \langle u, d\tau\rangle = 0 && \forall \tau \in V^{k-1} \\ &\langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle = \langle f, v\rangle && \forall v \in V^k \\ &\langle u, q\rangle = 0 && \forall q \in \mathfrak{H}^k. \end{aligned} \tag{2.4}$$

[1)] We choose these sign conventions to relate to the more concrete situations later on, which in turn is done to match [1, 3, 24]. This is, however, opposite the convention in Brüning and Lesch [8].

The first equation is the weak form of $\sigma = d^*u$, the second is the main equation (2.3) modified to account for a harmonic term so that a solution exists, and the third enforces uniqueness by requiring perpendicularity to the harmonic space. With these modifications, the problem is well-posed by considering the bilinear form (writing $\mathfrak{X}^k := V^{k-1} \times V^k \times \mathfrak{H}^k$) $B : \mathfrak{X}^k \times \mathfrak{X}^k \to \mathbb{R}$ defined by

$$B(\sigma, u, p; \tau, v, q) := \langle \sigma, \tau \rangle - \langle d\tau, u \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle - \langle u, q \rangle. \tag{2.5}$$

and linear functional $F \in (\mathfrak{X}^k)^*$ given by $F(\tau, v, q) = \langle f, v \rangle$. The form B is *not* coercive, but rather, for a closed Hilbert complex, satisfies an **inf-sup condition** [3, 4]: there exists $\gamma > 0$ (the **stability constant**) such that

$$\inf_{(\sigma,u,p)\neq 0} \sup_{(\tau,v,q)\neq 0} \frac{B(\sigma, u, p; \tau, v, q)}{\|(\sigma, u, p)\|_{\mathfrak{X}} \|(\tau, v, q)\|_{\mathfrak{X}}} =: \gamma > 0.$$

where we have defined a standard norm on products: $\|(\sigma, u, p)\|_{\mathfrak{X}} := \|\sigma\|_V + \|u\|_V + \|p\|$. This is sufficient to guarantee the well-posedness [4]. To summarize:

Theorem 2.3 ([3], Theorem 3.1). *The mixed problem* (2.4) *on a closed Hilbert complex* (W, d) *with domain* (V, d) *is well-posed: the bilinear form* B *satisfies the inf-sup condition with constant* γ*, so for any* $F \in (\mathfrak{X}^k)^*$*, there exists a unique solution* (σ, u, p) *to* (2.4)*, i.e.,* $B(\sigma, u, p; \tau, v, q) = F(\tau, v, q)$ *fo all* $(\tau, v, q) \in \mathfrak{X}^k$*, and moreover,*

$$\|(\sigma, u, p)\|_{\mathfrak{X}} \leq \gamma^{-1} \|F\|_{\mathfrak{X}^*}.$$

The ***stability constant*** γ^{-1} *depends only on the Poincaré constant.*

Note that the general theory (e.g., [4, 19]) guarantees a unique solution exists for *any* bounded linear functional $F \in (\mathfrak{X}^k)^*$, which in this case with product spaces, means that the problem is still well-posed when there are other nonzero linear functionals on the right sides of (2.4) besides $\langle f, v \rangle$. We shall need this result for parabolic problems, where we assume $P_{\mathfrak{H}} u \neq 0$.

2.2. Approximation of Hilbert Complexes

We now approximate solutions to the abstract mixed Hodge Laplacian problem. To do so, Arnold, Falk, and Winther [3] introduce finite-dimensional subspaces $V_h \subseteq V$ of the domain complex, such that the inclusion $i_h : V_h \hookrightarrow V$ is a morphism, i.e. $dV_h^k \subseteq V_h^{k+1}$. With the weak form (2.4), we formulate the Galerkin method by restricting to the subspaces:

$$\begin{aligned} &\langle \sigma_h, \tau \rangle - \langle u_h, d\tau \rangle = 0 && \forall \tau \in V_h^{k-1} \\ &\langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle = \langle f, v \rangle && \forall v \in V_h^k \\ &\langle u_h, q \rangle = 0 && \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{2.6}$$

We abbreviate by setting $\mathfrak{X}_h^k := V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$. We must also assume the existence of bounded, surjective, and idempotent (projection) morphisms $\pi_h : V \to V_h$. It is generally not the orthogonal projection, as that fails to commute with the differentials. As a projection, it gives the following **quasi-optimality** result:

$$\|u - \pi_h u\|_V = \inf_{v \in V_h} \|(I - \pi_h)(u - v)\|_V \leq \|I - \pi_h\| \inf_{v \in V_h} \|u - v\|_V.$$

The problem (2.6) is then well-posed, with a Poincaré constant given by $c_P\|\pi_h^k\|$, where c_P is the Poincaré constant for the continuous problem. This guarantees all the previous abstract results apply to this case. With this, we have the following error estimate:

Theorem 2.4 ([3], Theorem 3.9). *Let (V_h, d) be a family of subcomplexes of the domain complex (V, d) of a closed Hilbert complex, parametrized by h and admitting uniformly V-bounded cochain projections π_h, and let $(\sigma, u, p) \in \mathfrak{X}^k$ be the solution of the continuous problem and $(\sigma_h, u_h, p_h) \in \mathfrak{X}_h^k$ the corresponding discrete solution. Then the following error estimate holds:*

$$\begin{aligned}&\|(\sigma-\sigma_h, u-u_h, p-p_h)\|_{\mathfrak{X}} = \|\sigma-\sigma_h\|_V + \|u-u_h\|_V + \|p-p_h\| \\ \le& C\left(\inf_{\tau\in V_h^{k-1}}\|\sigma-\tau\|_V + \inf_{v\in V_h^k}\|u-v\|_V + \inf_{q\in V_h^k}\|p-q\|_V + \mu\inf_{v\in V_h^k}\|P_{\mathfrak{B}}u-v\|_V\right)\end{aligned} \tag{2.7}$$

with $\mu = \mu_h^k = \sup_{\substack{r\in\mathfrak{H}^k\\ \|r\|=1}}\left\|\left(I-\pi_h^k\right)r\right\|$, the operator norm of $I-\pi_h^k$ restricted to $\mathfrak{H}^k$.

Corollary 2.1. *If the V_h approximate V, that is, for all $u \in V$, $\inf_{v\in V_h}\|u-v\|_V \to 0$ as $h \to 0$, we have convergence of the approximations.*

In general, the harmonic spaces $\mathfrak{H}^k$ and $\mathfrak{H}_h^k$ do not coincide, but they are isomorphic under many circumstances we shall consider (namely, the spaces are isomorphic if for all harmonic forms $q \in \mathfrak{H}^k$, the error $\|q-\pi_h q\|$ is at most the norm $\|q\|$ itself [3, Theorem 3.4], and it *always* holds for the de Rham complex). For a quantitative estimate relating the two different kinds of harmonic forms, we have the following

Theorem 2.5 ([3], Theorem 3.5). *Let (V, d) be a bounded, closed Hilbert complex, (V_h, d) a Hilbert subcomplex, and π_h a bounded cochain projection. Then*

$$\|(I-P_{\mathfrak{H}_h})q\|_V \le \|(I-\pi_h^k)q\|_V, \quad \forall q\in\mathfrak{H}^k, \tag{2.8}$$

$$\|(I-P_{\mathfrak{H}})q\|_V \le \|(I-\pi_h^k)P_{\mathfrak{H}}q\|_V, \quad \forall q\in\mathfrak{H}_h^k. \tag{2.9}$$

2.3. Removing the Subcomplex Assumption: Variational Crimes

For geometric problems, it is essential to remove the requirement that the approximating complex V_h actually be subspaces of V. This is motivated by the example of approximating planar domains with curved boundaries by piecewise-linear approximations, resulting in finite element spaces that lie in a different function space [6]. Holst and Stern [24] extend the Arnold, Falk, Winther [3] framework by supposing that $i_h : V_h \hookrightarrow V$ is an injective morphism which is not necessarily inclusion; they also require projection morphisms $\pi_h : V \to V_h$ with the property $\pi_h \circ i_h = \mathrm{id}$, which replaces the idempotency requirement of the preceding case. To summarize, given (W, d) a Hilbert complex with domain complex (V, d), (W_h, d_h) another complex (whose inner product we denote $\langle\cdot,\cdot\rangle_h$) with domain complex (V_h, d_h), injective morphisms $i_h : W_h \hookrightarrow W$, and finally, projection morphisms $\pi_h : V \to V_h$. We then have the following generalized Galerkin problem:

$$\begin{aligned}&\langle\sigma_h,\tau_h\rangle_h - \langle u_h, d_h\tau_h\rangle_h = 0, && \forall\tau_h\in V_h^{k-1},\\ &\langle d_h\sigma_h, v_h\rangle_h + \langle d_h u_h, d_h v_h\rangle_h + \langle p_h, v_h\rangle_h = \langle f_h, v_h\rangle_h, && \forall v_h\in V_h^k,\\ &\langle u_h, q_h\rangle_h = 0, && \forall q_h\in\mathfrak{H}_h^k,\end{aligned} \tag{2.10}$$

where f_h is some interpolation of the given data f into the space W_h (we will discuss various choices of this operator later). This gives us a bilinear form

$$\begin{aligned} &B_h(\sigma_h, u_h, p_h; \tau_h, v_h, q_h) \\ :=&\langle \sigma_h, \tau_h\rangle_h - \langle u_h, d_h\tau_h\rangle_h + \langle d_h\sigma_h, v_h\rangle_h + \langle d_h u_h, d_h v_h\rangle_h + \langle p_h, v_h\rangle_h - \langle u_h, q_h\rangle_h. \end{aligned} \tag{2.11}$$

This problem is well-posed, which again follows from the abstract theory as long as the complex is closed, and there is a corresponding Poincaré inequality:

Theorem 2.6 ([24], Thm. 3.5 and Cor. 3.6). *Let (V, d) and (V_h, d_h) be bounded closed Hilbert complexes, with morphisms $i_h : V_h \hookrightarrow V$ and $\pi_h : V \to V_h$ such that $\pi_h \circ i_h = \mathrm{id}$. Then for all $v_h \in \mathfrak{Z}_h^{k\perp}$, we have*

$$\|v_h\|_{V_h} \le c_P \left\|\pi_h^k\right\| \left\|i_h^{k+1}\right\| \|d_h v_h\|_{V_h},$$

where c_P is the Poincaré constant corresponding to the continuous problem. If (V, d) and (V_h, d_h) are the domain complexes of closed complexes (W, d) and (W_h, d_h), then $\|d_h v_h\|_{V_h}$ is simply $\|d_h v_h\|_h$ (since it is the graph norm and $d^2 = 0$).

In other words, the norm of the injective morphisms i_h also contributes to the stability constant for this discrete problem. Analysis of this method results in two additional error terms (along with now having to explicitly reference the injective morphisms i_h which may no longer be inclusions), due to the inner products in the space V_h no longer necessarily being the restriction of that in V: the need to approximate the data f, and the failure of the morphisms i_h to be unitary.

Theorem 2.7 ([24], Corollary 3.11). *Let (V, d) be the domain complex of a closed Hilbert complex (W, d), and (V_h, d_h) the domain complex of (W_h, d_h) with morphisms $i_h : W_h \to W$ and $\pi_h : V \to V_h$ as above. Then if we have solutions (σ, u, p) and (σ_h, u_h, p_h) to (2.4) and (2.10) respectively, the following error estimate holds:*

$$\begin{aligned} &\|\sigma - i_h\sigma_h\|_V + \|u - i_h u_h\|_V + \|p - i_h p_h\| \\ \le& C\Bigg(\inf_{\tau\in i_h V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v\in i_h V_h^k} \|u - v\|_V + \inf_{q\in i_h V_h^k} \|p - q\|_V \\ &\qquad + \mu \inf_{v\in i_h V_h^k} \|P_{\mathfrak{B}} u - v\|_V + \|f_h - i_h^* f\|_h + \|I - J_h\| \, \|f\| \Bigg), \end{aligned} \tag{2.12}$$

where $J_h = i_h^* i_h$, and $\mu = \mu_h^k = \sup\limits_{\substack{r\in\mathfrak{H}^k \\ \|r\|=1}} \left\|\left(I - i_h^k \pi_h^k\right) r\right\|$.

The extra terms (in the third line of the inequality) are analogous to the terms described in the Strang lemmas [6, §III.1]. The main idea of the proof of Theorem 2.7 (which we will recall in more detail below, because we will need to prove a generalization of it as part of our main results) is to form an intermediate complex by pulling the inner products in the complex (W, d) back to (W_h, d_h) back by i_h, construct a solution to the problem there, and compare that solution with the solution we want. This modified inner product does not coincide with the given one on W_h precisely when i_h is not unitary:

$$\langle v, w\rangle_{i_h^* W} = \langle i_h v, i_h w\rangle_h = \langle i_h^* i_h v, w\rangle_h = \langle J_h v, w\rangle_h.$$

Unitarity is simply the condition $J_h = I$. The complex W_h with the modified inner product now may be identified with a true subcomplex of W, for which the theory of [3] directly applies, yielding a solution $(\sigma'_h, u'_h, p'_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}'^k_h$, where $\mathfrak{H}'^k_h$ is the discrete harmonic space associated to the space with the modified inner product. This generally does not coincide with the discrete harmonic space $\mathfrak{H}^k_h$, since the discrete codifferential $d^{*\prime}_h$ in that case is defined to be the adjoint with respect to the modified inner product, yielding a different Hodge decomposition. The estimate of $\|i_h\sigma'_h - \sigma\|_V + \|i_h u'_h - u\|_V + \|i_h p'_h - p\|$ then proceeds directly from the preceding theory for subcomplexes (2.7). The variational crimes, on the other hand, arise from comparing the solution (σ_h, u_h, p_h) with (σ'_h, u'_h, p'_h). Finally, the error estimate (2.12) proceeds by the triangle inequality (and the boundedness of the morphisms i_h).

2.4. Elliptic Error Estimates for a Nonzero Harmonic Part

Our objective in the remainder of this section is to prove one of our main results, a generalization of Theorem 2.7 which allows the possibility of the solution u having a nonzero harmonic part w. The first lemma extends the problem in the subcomplex case; then the following theorem extends that result for the case with variational crimes.

Lemma 2.1. *Theorem* 2.4 *continues to apply when, in the continuous problem, we replace the third equation of* (2.4) $\langle u, q\rangle = 0$ *with* $\langle u, q\rangle = \langle w, q\rangle$ *where* $w \in \mathfrak{H}^k$ *is prescribed.*

In other words, if in the continuous problem, instead of enforcing orthogonality of the solution u to the harmonic forms ($P_{\mathfrak{H}} u = 0$), we instead enforce the condition $P_{\mathfrak{H}} u = w$, both the discrete and continuous versions of the problem are still well-posed, and the same error estimates hold.

Proof. We closely follow the proof, in [3], of Theorem 2.4 above, noting where the modifications must occur. Let B be the bounded bilinear form (2.5); then (σ, u, p) satisfies, for all $(\tau_h, v_h, q_h) \in \mathfrak{X}^k_h$,

$$B(\sigma, u, p; \tau_h, v_h, q_h) = \langle f, v_h\rangle - \langle u, q_h\rangle.$$

We V-orthogonally project (σ, u, p) in each factor to $(\tau, v, q) \in \mathfrak{X}^k_h$. Then for any $(\tau_h, v_h, q_h) \in \mathfrak{X}^k_h$,

$$\begin{aligned}
&B(\sigma_h - \tau, u_h - v, p_h - q; \tau_h, v_h, q_h)\\
=&B(\sigma - \tau, u - v, p - q; \tau_h, v_h, q_h) + \langle u, q_h\rangle - \langle w, q_h\rangle\\
=&B(\sigma - \tau, u - v, p - q; \tau_h, v_h, q_h) + \langle P_{\mathfrak{H}_h}(u - w), q_h\rangle\\
\le&C\Big(\|\sigma - \tau\|_V + \|u - v\|_V + \|p - q\| + \|P_{\mathfrak{H}_h}(u - w)\|)(\|\tau_h\|_V + \|v_h\|_V + \|q_h\|\Big).
\end{aligned} \tag{2.13}$$

Noticing that the factor $p_h - q$ in the bilinear form above is in the original domain $\mathfrak{H}^k_h$, we can now choose the appropriate (τ_h, v_h, q_h) that verifies the inf-sup condition of B:

$$\begin{aligned}
&B(\sigma_h - \tau, u_h - v, p_h - q; \tau_h, v_h, q_h)\\
\ge&\gamma\Big(\|\sigma_h - \tau\|_V + \|u_h - v\|_V + \|p_h - q\|)(\|\tau_h\|_V + \|v_h\|_V + \|q_h\|\Big).
\end{aligned}$$

Comparing this to (2.13) above, we may cancel the common factor, and divide by γ to arrive at

$$\begin{aligned}
&\|\sigma_h - \tau\|_V + \|u_h - v\|_V + \|p_h - q\|\\
\le&C\gamma^{-1}\Big(\|\sigma - \tau\|_V + \|u - v\|_V + \|p - q\| + \|P_{\mathfrak{H}_h}(u - w)\|\Big).
\end{aligned} \tag{2.14}$$

This differs (aside from the notation) from [3] in that we now have, rather than $P_{\mathfrak{H}_h}u$, instead $P_{\mathfrak{H}_h}(u-w)$, with the harmonic part subtracted off. Removing the harmonic part allows us to continue as in [3]: the Hodge decomposition $u-w=u-P_{\mathfrak{H}}u$ consists only of coboundary and perpendicular terms $u_{\mathfrak{B}}+u_{\perp}\in\mathfrak{B}^k\oplus\mathfrak{Z}^{k\perp}$. With $\mathfrak{H}_h^k$ contained in $\mathfrak{Z}^k$, it follows $P_{\mathfrak{H}_h}u_{\perp}=0$, and $P_{\mathfrak{H}_h}\pi_h u_{\mathfrak{B}}=0$. Also, $(I-\pi_h)u_{\mathfrak{B}}$ is perpendicular to $\mathfrak{H}^k$. Therefore, for all $q\in\mathfrak{H}_h^k$,

$$\begin{aligned}&\langle P_{\mathfrak{H}_h}(u-P_{\mathfrak{H}}u),q\rangle=\langle P_{\mathfrak{H}_h}u_{\mathfrak{B}},q\rangle=\langle P_{\mathfrak{H}_h}(u_{\mathfrak{B}}-\pi_h u_{\mathfrak{B}}),q\rangle\\ =&\langle u_{\mathfrak{B}}-\pi_h u_{\mathfrak{B}},q\rangle=\langle u_{\mathfrak{B}}-\pi_h u_{\mathfrak{B}},(I-P_{\mathfrak{H}})q\rangle.\end{aligned}$$

Now, setting

$$q=\frac{P_{\mathfrak{H}_h}(u-P_{\mathfrak{H}}u)}{\|P_{\mathfrak{H}_h}(u-P_{\mathfrak{H}}u)\|}\in\mathfrak{H}_h^k,$$

we have

$$\begin{aligned}&\|P_{\mathfrak{H}_h}(u-P_{\mathfrak{H}}u)\|=\langle P_{\mathfrak{H}_h}(u-P_{\mathfrak{H}}u),q\rangle=\langle u_{\mathfrak{B}}-\pi_h u_{\mathfrak{B}},(I-P_{\mathfrak{H}})q\rangle\\ \le&\|u_{\mathfrak{B}}-\pi_h u_{\mathfrak{B}}\|\,\|(I-P_{\mathfrak{H}})q\|\le C\|(I-P_{\mathfrak{H}})q\|\inf_{v\in V_h^k}\|u_{\mathfrak{B}}-v\|_V.\end{aligned}$$

Finally, by the second estimate of Theorem 2.5 above, we can bound $\|(I-P_{\mathfrak{H}})q\|$ by $\|(I-\pi_h)P_{\mathfrak{H}}q\|$, giving us

$$\|(I-P_{\mathfrak{H}})q\|\le\|(I-\pi_h)P_{\mathfrak{H}}q\|\le\sup_{\|r\|=1 r\in\mathfrak{H}^k}\|(I-\pi_h)r\|\,\|P_{\mathfrak{H}}q\|\le\mu.$$

From the triangle inequality, we derive the estimate

$$\begin{aligned}&\|\sigma-\sigma_h\|_V+\|u-u_h\|_V+\|p-p_h\|\\ \le&\|\sigma-\tau\|_V+\|u-v\|_V+\|p-q\|+\|\tau-\sigma_h\|_V+\|u_h-v\|_V+\|q-p_h\|\\ \le&(1+C\gamma^{-1})\Big(\|\sigma-\tau\|_V+\|u-v\|_V+\|p-q\|+\mu\inf_{v\in i_hV_k^h}\|P_{\mathfrak{B}}u-v\|_V\Big).\end{aligned}$$

Using best approximation property of orthogonal projections, we can express the remaining terms with the infima, and this gives the result. □

We are now ready to prove our main elliptic error estimate, an extension of Theorem 2.7. To set things up, consider the problems (2.4) and (2.10) but instead with now with prescribed, possibly nonzero harmonic part w. Given $f\in W^k$ and $w\in\mathfrak{H}^k$, we seek $(\sigma,u,p)\in\mathfrak{X}^k$ such that

$$\begin{aligned}&\langle\sigma,\tau\rangle-\langle u,d\tau\rangle=0 && \forall\tau\in V^{k-1}\\ &\langle d\sigma,v\rangle+\langle du,dv\rangle+\langle p,v\rangle=\langle f,v\rangle && \forall v\in V^k\\ &\langle u,q\rangle=\langle w,q\rangle && \forall q\in\mathfrak{H}^k.\end{aligned}\tag{2.15}$$

By Theorem 2.3, this problem is well-posed, and $\|(\sigma,u,p)\|_{\mathfrak{X}}\le c(\|f\|+\|w\|)$, with c depending only on the Poincaré constant. Now, consider the discrete problem, with $f_h,w_h\in V_h^k$:

$$\begin{aligned}&\langle\sigma_h,\tau_h\rangle_h-\langle u_h,d_h\tau_h\rangle_h=0 && \forall\tau_h\in V_h^{k-1}\\ &\langle d_h\sigma_h,v_h\rangle_h+\langle d_hu_h,d_hv_h\rangle_h+\langle p_h,v_h\rangle_h=\langle f_h,v_h\rangle_h && \forall v_h\in V_h^k\\ &\langle u_h,q_h\rangle_h=\langle w_h,q_h\rangle_h && \forall q_h\in\mathfrak{H}_h^k.\end{aligned}\tag{2.16}$$

This problem is also well-posed, with the modified Poincaré constant in Theorem 2.6.

Theorem 2.8 (Extension of elliptic error estimates, allowing for a harmonic part). *Let f, w, (σ, u, p), and (σ_h, u_h, p_h) be as above. Then we have the following generalization of the error estimate* (2.12) *above:*

$$\begin{aligned}
&\|\sigma - i_h\sigma_h\|_V + \|u - i_h u_h\|_V + \|p - i_h p_h\| \\
\leq &C\Big(\inf_{\tau\in i_h V_h^{k-1}} \|\sigma-\tau\|_V + \inf_{v\in i_h V_h^k} \|u-v\|_V + \inf_{q\in i_h V_h^k} \|p-q\|_V + \mu \inf_{v\in i_h V_h^k} \|P_{\mathfrak{B}} u - v\|_V \\
&+ \inf_{\xi\in i_h V_h^k} \|w-\xi\|_V + \|f_h - i_h^* f\|_h + \|w_h - i_h^* w\|_h + \|I - J_h\| \left(\|f\| + \|w\|\right)\Big), \qquad (2.17)
\end{aligned}$$

where, as before, $J_h = i_h^* i_h$, and $\mu = \mu_h^k = \sup_{r\in\mathfrak{H}^k \|r\|=1} \left\| \left(I - i_h^k \pi_h^k\right) r \right\|$.

We see that three new error terms arise from the approximation of the harmonic part, one being the data interpolation error (but measured in the V_h-norm, which partially captures how d fails to commute with i_h^* and how w_h may not necessarily be a discrete harmonic form), another best approximation term, and finally another term from the non-unitarity. As mentioned in the introduction, and following [24], we often shall take $f_h = \Pi_h f$, where Π_h is some family of linear interpolation operators with $\Pi_h \circ i_h = \mathrm{id}$. Another seemingly obvious choice is i_h^* itself (thus making those corresponding error terms zero), but as mentioned in [24], this can be difficult to compute, so we do not restrict ourselves to this case. Various choices of interpolation will be crucial in deciding which estimates to make in the parabolic problem. We split the proof of this theorem into two parts, the first of which derives the quantities on the second line of (2.17), and the second part, we derive the quantities on the third line of (2.17). Generally, we follow the pattern of proof in [3, Theorem 3.9] and [24, Theorem 3.10], noting the necessary modifications, as well as a similar technique given for the improved error estimates by Arnold and Chen [1].

Proof. [First part of the proof of Theorem 2.8] First, following Holst and Stern [24] as above, we construct the complex W_h but with the modified inner product $\langle J_h \cdot, \cdot\rangle$ (the associated domain complex V_h remains the same). This gives us a discrete Hodge decomposition with another type of orthogonality and corresponding discrete harmonic forms and orthogonal complement (due to a different adjoint $d_h^{*\prime}$):

$$V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h'^k \oplus \mathfrak{Z}_h^{k\perp\prime}$$

(generally, primed objects will represent the corresponding objects defined with the modified inner product; the discrete coboundaries are in fact the same as before, because d and d_h do not depend on the choice of inner product). The main complications arise in having to keep careful track of the different harmonic forms involved; the additional error terms, in some sense, measure this difference. We then define, as in [24], the intermediate solution $(\sigma_h', u_h', p_h') \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h'^k$ (which we abbreviate as $\mathfrak{X}_h'^k$):

$$\begin{aligned}
&\langle J_h\sigma_h', \tau_h\rangle_h - \langle J_h u_h', d_h\tau_h\rangle_h = 0 && \forall \tau_h \in V_h^{k-1} \\
&\langle J_h d_h\sigma_h', v_h\rangle_h + \langle J_h d_h u_h', d_h v_h\rangle_h + \langle J_h p_h', v_h\rangle_h = \langle i_h^* f, v_h\rangle_h && \forall v_h \in V_h^k \qquad (2.18)\\
&\langle J_h u_h', q_h'\rangle_h = \langle i_h^* w, q_h'\rangle_h && \forall q_h' \in \mathfrak{H}_h'^k,
\end{aligned}$$

and the corresponding bilinear form $B_h' : \mathfrak{X}_h' \times \mathfrak{X}_h' \to \mathbb{R}$ given by

$$\begin{aligned}
B_h'(\sigma_h', u_h', p_h'; \tau_h, v_h, q_h') := &\langle J_h\sigma_h', \tau_h\rangle_h - \langle J_h u_h', d_h\tau_h\rangle_h \\
&+ \langle J_h d_h\sigma_h', v_h\rangle_h + \langle J_h d_h u_h', d_h v_h\rangle_h + \langle J_h p_h', v_h\rangle_h - \langle J_h u_h', q_h'\rangle_h. \qquad (2.19)
\end{aligned}$$

This satisfies the inf-sup condition with Poincaré constant $c_P\|\pi_h\|$. Note that we will need to extend all the bilinear forms B_h, and B'_h in the last factor to all of V_h^k in order to compare the two, since they are initially only defined on the respective, *differing* harmonic form spaces; we must remember to invoke the inf-sup condition only when using the non-extended versions. The idea is, again, to use the triangle inequality:

$$\|\sigma - i_h\sigma_h\|_V + \|\tau - i_h\tau_h\|_V + \|p - i_h p_h\| \tag{2.20}$$

$$\leq \|\sigma - i_h\sigma'_h\|_V + \|\tau - i_h\tau'_h\|_V + \|p - i_h p'_h\| \tag{2.21}$$

$$+ \|i_h(\sigma'_h - \sigma_h)\|_V + \|i_h(\tau'_h - \tau_h)\|_V + \|i_h(p'_h - p_h)\|. \tag{2.22}$$

These quantities can be estimated using only geometric properties of the domain; we never actually compute (σ'_h, u'_h, p'_h). To estimate the term (2.21) (which we shall refer to as the PDE approximation term, whereas (2.22) will be called variational crimes), we recall that i_h is an isometry of W_h with the modified inner product to its image, which is a subcomplex.

Thus, Lemma 2.1 above applies, with the approximation $(i_h\sigma'_h, i_h u'_h, i_h p'_h)$ on the identified subcomplex $i_h\mathfrak{X}_h'^k$. This gives us the terms on the second line of (2.17).

To finish our main proof, we need to consider the variational crimes (2.22). Since the maps i_h are bounded, and we eventually absorb their norms into the constant C above, it suffices to consider $\|\sigma_h - \sigma'_h\|_{V_h} + \|u_h - u'_h\|_{V_h} + \|p_h - p'_h\|_h$, which we shall state as a separate theorem.

Theorem 2.9. *Let $(\sigma_h, u_h, p_h) \in \mathfrak{X}_h^k$ be a solution to* (2.16)*, $(\sigma'_h, u'_h, p'_h) \in \mathfrak{X}_h'^k$ a solution to* (2.18)*, and $w = P_{\mathfrak{H}}u$, the prescribed harmonic part of the continuous problem. Then*

$$\begin{aligned}&\|\sigma_h - \sigma'_h\|_{V_h} + \|u_h - u'_h\|_{V_h} + \|p_h - p'_h\|_h\\ \leq& C\Big(\|f_h - i_h^* f\|_h + \|w_h - i_h^* w\|_{V_h} + \|I - J_h\|(\|f\| + \|w\|) + \inf_{\xi \in i_h V_h^k}\|w - \xi\|_V\Big),\end{aligned} \tag{2.23}$$

i.e., they are bounded by the terms on the third line in (2.17)*.*

Proof. [Proof of Theorem 2.9 and second part of the proof of Theorem 2.8] We follow the proof of Holst and Stern [24, Theorem 3.10] and note the modifications. Let (τ, v, q) and $(\tau_h, v_h, q_h) \in \mathfrak{X}_h^k$. Consider the bilinear form B_h, (2.11) above, and write

$$\begin{aligned}&B_h(\sigma_h - \tau, u_h - v, p_h - q; \tau_h, v_h, w_h)\\ =&B_h(\sigma_h - \sigma'_h, u_h - u'_h, p_h - p'_h; \tau_h, v_h, q_h) + B_h(\sigma'_h - \tau, u'_h - v, p'_h - q; \tau_h, v_h, q_h).\end{aligned}$$

We then have, recalling the modified bilinear form B'_h, (2.19) above, and extending it in the last factors to all of V_h^k,

$$\begin{aligned}&B_h(\sigma'_h, u'_h, p'_h; \tau_h, v_h, q_h)\\ =&B'_h(\sigma'_h, u'_h, p'_h; \tau_h, v_h, q_h) + \langle (I - J_h)\sigma'_h, \tau_h\rangle_h - \langle (I - J_h)u'_h, d_h\tau_h\rangle_h + \langle (I - J_h)d_h\sigma'_h, v_h\rangle_h\\ &+ \langle (I - J_h)d_h u'_h, d_h v_h\rangle_h + \langle (I - J_h)p'_h, v_h\rangle_h - \langle (I - J_h)u'_h, q_h\rangle_h.\end{aligned}$$

Substituting the respective solutions (2.18) and (2.16) (and noting the slight discrepancy in the use of different harmonic forms), we have

$$\begin{aligned}B'_h(\sigma'_h, u'_h, p'_h; \tau_h, v_h, q_h) &= \langle i_h^* f, v_h\rangle_h - \langle J_h u'_h, q_h\rangle_h\\ B_h(\sigma_h, u_h, p_h; \tau_h, v_h, q_h) &= \langle f_h, v_h\rangle_h - \langle w_h, q_h\rangle_h.\end{aligned}$$

Consequently,

$$\begin{aligned}
&B_h(\sigma_h - \sigma_h', u_h - u_h', p_h - p_h'; \tau_h, v_h, q_h)\\
=&\langle f_h - i_h^* f, v_h\rangle_h + \langle u_h', q_h\rangle_h - \langle w_h, q_h\rangle_h - \langle (I - J_h)\sigma_h', \tau_h\rangle_h + \langle (I - J_h)u_h', d_h\tau_h\rangle_h\\
&- \langle (I - J_h)d_h\sigma_h', v_h\rangle_h - \langle (I - J_h)d_h u_h', d_h v_h\rangle_h - \langle (I - J_h)p_h', v_h\rangle_h.
\end{aligned}$$

As before, we bound the form above and below. For the upper bound, using Cauchy-Schwarz to estimate the extra inner product terms, we arrive at

$$\begin{aligned}
&B_h(\sigma_h - \tau, u_h - v, p_h - q; \tau_h, v_h, q_h)\\
\le& C\Big(\|f_h - i_h^* f\|_h + \|P_{\mathfrak{H}_h}(u_h' - w_h)\|_h + \|I - J_h\|(\|\sigma_h'\|_{V_h} + \|u_h'\|_{V_h} + \|p_h'\|_h)\\
&+ \|\sigma_h' - \tau\|_{V_h} + \|u_h' - v\|_{V_h} + \|p_h' - q\|_h)(\|\tau_h\|_{V_h} + \|v_h\|_{V_h} + \|q_h\|_h\Big).
\end{aligned}$$

For the lower bound, we again choose $(\tau_h, \sigma_h, q_h) \in \mathfrak{X}_h^k$ to verify the inf-sup condition this time for B_h:

$$\begin{aligned}
&B_h(\sigma_h - \tau, u_h - v, p_h - q; \tau_h, v_h, q_h)\\
\ge&\gamma_h\Big(\|\sigma_h - \tau\|_{V_h} + \|u_h - v\|_{V_h} + \|p_h - q\|_h)(\|\tau_h\|_{V_h} + \|v_h\|_{V_h} + \|q_h\|_h\Big),
\end{aligned}$$

γ_h depending only on the Poincaré constant $c_P\|i_h\|\,\|\pi_h\|$. Comparing with the upper bound and dividing out the common factor as before, this leads to:

$$\begin{aligned}
&\|\sigma_h - \tau\|_{V_h} + \|u_h - v\|_{V_h} + \|p_h - q\|_h\\
\le& C\gamma_h^{-1}\Big(\|f_h - i_h^* f\|_h + \|P_{\mathfrak{H}_h}(u_h' - w_h)\|_h + \|I - J_h\|(\|\sigma_h'\|_{V_h} + \|u_h'\|_{V_h} + \|p_h'\|_h)\\
&+ \|\sigma_h' - \tau\|_{V_h} + \|u_h' - v\|_{V_h} + \|p_h' - q\|_h\Big).
\end{aligned}$$

Choosing $(\tau, v, q) = (\sigma_h', u_h', P_{\mathfrak{H}_h} p_h')$, applying the triangle inequality with p_h' to account for the mismatch in the harmonic spaces, and using the well-posedness of the continuous problem (2.18),

$$\begin{aligned}
&\|\sigma_h - \sigma_h'\|_{V_h} + \|u_h - u_h'\|_{V_h} + \|p_h - p_h'\|_h\\
\le& C\Big(\|f_h - i_h^* f\|_h + \|P_{\mathfrak{H}_h}(u_h' - w_h)\|_h + \|I - J_h\|(\|f\| + \|w\|) + \|p_h' - q\|_h\Big).
\end{aligned}$$

This differs from [24] in that we have the bound in terms of $\|f\| + \|w\|$, and that we must estimate $\|P_{\mathfrak{H}_h}(u_h' - w_h)\|_h$ rather than $\|P_{\mathfrak{H}_h} u_h'\|_h$ alone. First, we use the modified Hodge decomposition to uniquely write u_h' as $u_{\mathfrak{B}}' + P_{\mathfrak{H}_h'} u_h' + u_\perp'$ with $u_{\mathfrak{B}}' \in \mathfrak{B}_h^k$ and $u_\perp' \in \mathfrak{Z}_h^{k\perp\prime}$, and

$$\|P_{\mathfrak{H}_h}(u_h' - w_h)\|_h \le \|P_{\mathfrak{H}_h}(u_{\mathfrak{B}}' + u_\perp')\|_h + \|P_{\mathfrak{H}_h}(P_{\mathfrak{H}_h'} u_h' - w_h)\|_h.$$

(The projection $P_{\mathfrak{H}_h'}$ is with respect to the *modified* inner product). For the first term, we proceed as in [24]: we have $P_{\mathfrak{H}_h} u_{\mathfrak{B}}' = 0$ since coboundaries are independent of inner products. Now $u_\perp' \in \mathfrak{Z}_h^{k\perp\prime}$ so, using J_h to express it in terms of V-orthogonality, we have $J_h u_\perp' \perp \mathfrak{Z}_h^k$, and thus $P_{\mathfrak{H}_h} J_h u_\perp' = 0$. Therefore, we have

$$\|P_{\mathfrak{H}_h}(u_{\mathfrak{B}}' + u_\perp')\|_h = \|P_{\mathfrak{H}_h} u_\perp'\|_h = \|P_{\mathfrak{H}_h}(I - J_h)u_\perp'\|_h \le C\|I - J_h\|(\|f\| + \|w\|).$$

For the p'_h term, this also proceeds as in [24] (except for, of course, the extra $\|w\|$ term): using the (unmodified) discrete Hodge decomposition, we have $p'_h = P_{\mathfrak{B}_h} p'_h + P_{\mathfrak{H}_h} p'_h = P_{\mathfrak{B}_h} p'_h + q$. Since $p'_h \in \mathfrak{H}'^k_h$, a similar argument gives $J_h p'_h \perp \mathfrak{B}^k_h$, so $P_{\mathfrak{B}_h} J_h p'_h = 0$ and

$$\|p'_h - q\|_h = \|P_{\mathfrak{B}_h} p'_h\|_h = \|P_{\mathfrak{B}_h}(I - J_h)p'_h\|_h \leq C\|I - J_h\|(\|f\| + \|w\|).$$

Finally, we must consider the term $\|P_{\mathfrak{H}_h}(P_{\mathfrak{H}'_h} u'_h - w_h)\|_h$. We wish to express u'_h in terms of $i^*_h w$, as we did with f_h and $i^*_h f$, but this is more difficult, since $i^*_h w$ and w_h operate as linear functionals on different harmonic spaces. We recall the third equation of (2.18):

$$\langle J_h u'_h, q' \rangle_h = \langle i^*_h w, q' \rangle_h = \langle J_h (J_h^{-1} i^*_h w), q' \rangle_h$$

which, writing $i^+_h = J_h^{-1} i^*_h$ (the Moore-Penrose pseudoinverse of i_h), therefore says $P_{\mathfrak{H}'_h} u'_h = P_{\mathfrak{H}'_h} i^+_h w$. This enables us to properly work with the modified orthogonal projection $P_{\mathfrak{H}'_h}$. Note that i^+_h is an isometry of the subspace $i_h W_h$ to W_h *with the modified inner product*, so we have

$$P_{\mathfrak{H}'_h} i^+_h w = i^+_h P_{i_h \mathfrak{H}'_h} w.$$

where now $P_{i_h \mathfrak{H}'_h}$ is the orthogonal projection onto the identified image harmonic space. Then, using the triangle inequality with both $i^+_h w$ and $i^*_h w$,

$$\begin{aligned}
&\|P_{\mathfrak{H}_h}(P_{\mathfrak{H}'_h} u'_h - w_h)\|_h \\
\leq &\left\| P_{\mathfrak{H}_h} \left(P_{\mathfrak{H}'_h} i^+_h w - i^+_h w \right) \right\|_h + \|P_{\mathfrak{H}_h}(J_h^{-1} i^*_h w - i^*_h w)\|_h + \|P_{\mathfrak{H}_h}(i^*_h w - w_h)\|_h \\
\leq &\|P_{\mathfrak{H}_h}\| \left(\|i^+_h\| \left\| \left(I - P_{i_h \mathfrak{H}'_h} \right) w \right\| + \|J_h^{-1}\| \, \|I - J_h\| \, \|i^*_h w\|_h + \|i^*_h w - w_h\|_h \right) \\
\leq &C \left(\left\| \left(I - P_{i_h \mathfrak{H}'_h} \right) w \right\| + \|I - J_h\| \, \|w\| + \|i^*_h w - w_h\|_h \right).
\end{aligned}$$

The last term is the data approximation error for w, and the second term combines with the previous errors that reflect the non-unitarity of the operator. So, all that remains is to estimate the first term. Since it is in the subcomplex $i_h W_h$, the first estimate of Theorem 2.5 applies:

$$\left\| \left(I - P_{i_h \mathfrak{H}'_h} \right) w \right\| \leq \|(I - \pi'_h) w\| \leq C \inf_{\xi \in i_h V^k_h} \|w - \xi\|_V, \tag{2.24}$$

by quasi-optimality.

To summarize, we proved our main theorem, Theorem 2.8, by defining an intermediate solution on a modified complex that we identify with a subcomplex, and analyzing the result via the Arnold, Falk, and Winther [3] framework. That theorem holds, with the estimate unchanged, though now u and u_h no longer are perpendicular to their respective harmonic spaces. The place where the extra terms all show up is in the variational crimes. In the process of arriving at a term that looks like $i^*_h w - w_h$, working with the different harmonic forms produces two more non-unitarity terms $\|I - J_h\|(\|f\| + \|w\|)$, and finally, using Theorem 2.5 yields a direct estimate of how w fails to be a modified discrete harmonic form, giving the last best approximation term $\inf_{\xi \in i_h V^k_h} \|w - \xi\|_V$. □

We also note for future reference that in spaces where we have improved error estimates (which means π_h are W-bounded maps) that we can replace that last V-norm in (2.24) to be the W-inner product. Finally, we remark that, for a certain types of data interpolation (i.e. $f_h = \Pi_h f$ and $w_h = \Pi_h w$ where Π_h is the family of interpolation operators mentioned in the introduction and in the remarks following Theorem 2.8), the errors $\|f_h - i^*_h f\|$ and $\|w_h - i^*_h w\|$ can be rewritten in terms of the other errors and another best approximation term:

Theorem 2.10 ([24], Theorem 3.12). *If $\Pi_h : W^k \to W_h^k$ is a family of linear projections uniformly bounded with respect to h, then for all $f \in W^k$,*

$$\|\Pi_h f - i_h^* f\| \leq C\Big(\|I - J_h\| \, \|f\| + \inf_{\phi \in i_h W_h^k} \|f - \phi\|\Big). \tag{2.25}$$

This will be useful for us in our examples.

3. Abstract Evolution Problems

In order to solve and approximate linear evolution problems, we introduce the framework of Bochner spaces (also following Gillette, Holst, and Zhu [22]), which realizes time-dependent functions as curves in Banach spaces (which will correspond to spaces of spatially-dependent functions in our problem). We follow mostly [33] and [19] for this material.

3.1. Overview of Bochner Spaces and Abstract Evolution Problems

Let X be a Banach space and $I := [0, T]$ an interval in $\mathbb{R}$ with $T > 0$. We define

$$C(I, X) := \{u : I \to X \mid u \text{ bounded and continuous}\}.$$

In analogy to spaces of continuous, real-valued functions, we define a supremum norm on $C(I, X)$, making $C(I, X)$ into a Banach space:

$$\|u\|_{C(I,X)} := \sup_{t \in I} \|u(t)\|_X.$$

We will of course need to deal with norms other than the supremum norm, which motivates us to define **Bochner spaces**: to define $L^p(I, X)$, we complete $C(I, X)$ with the norm

$$\|u\|_{L^p(I,X)} := \left(\int_I \|u(t)\|_X^p dt\right)^{1/p}.$$

Similarly, we have the space $H^1(I, X)$, the completion of $C^1(I, X)$ with the norm

$$\|u\|_{H^1(I,X)} := \left(\int_I \|u(t)\|_X^2 + \left\|\frac{d}{dt}u(t)\right\|_X^2 dt\right)^{1/2}.$$

There are methods of formulating this in a more measure-theoretic way ([19, Appendix E]), considering Lebesgue-measurable subsets of I.

As mentioned before, for our purposes, X will be some space of spatially-dependent functions, and the time-dependence is captured as being a curve in this function space. Usually, X will be a space in some Hilbert complex, such as $L^2\Omega^k(M)$ or $H^s\Omega^k(M)$ where the forms are defined over a Riemannian manifold M.

We introduce this framework in order to be able to formulate parabolic problems more generally. It turns out to be useful to consider the concept of *rigged Hilbert space* or *Gelfand triple*, which consists of a triple of separable Banach spaces

$$V \subseteq H \subseteq V^*$$

such that V is continuously and densely embedded in H. For example, if (V, d) is the domain complex of some Hilbert complex (W, d), setting $V = V^k$ and $H = W^k$ works, as well as

various combinations of their products (so that we can use mixed formulations). H is also continuously embedded in V^*. We note carefully that the standard isomorphism (given by the Riesz representation theorem) between V and V^* is generally *not* the composition of the inclusions.

Given $A \in C(I, \mathcal{L}(V, V^*))$, a time-dependent linear operator, we would like to consider the abstract version of linear, parabolic problems:

$$u_t = A(t)u + f(t) \tag{3.1}$$

$$u(0) = u_0. \tag{3.2}$$

Toward this end, given A as above, we define the bilinear form

$$a(t, u, v) := \langle -A(t)u, v \rangle, \tag{3.3}$$

for $(t, u, v) \in \mathbb{R} \times V \times V$. As in elliptic problems, we need a to satisfy a kind of coercivity condition, Gårding's Inequality:

$$a(t, u, u) \geq c_1 \|u\|_V^2 - c_2 \|u\|_H^2, \tag{3.4}$$

with c_1, c_2 constants independent of $t \in I$. This is not as strong a condition as ellipticity. With this, the abstract parabolic problem is well-posed:

Theorem 3.1 (Existence and Uniqueness, [33], Theorem 11.3). *Let $f \in L^2(I, V^*)$ and $u_0 \in H$, and a the time-dependent quadratic form in* (3.3)*. Suppose* (3.4) *holds. Then the abstract parabolic problem* (3.1) *has a unique solution*

$$u \in L^2(I, V) \cap H^1(I, V^*).$$

Moreover, the Sobolev embedding theorem implies $u \in C(I, H)$, which allows us to unambiguously evaluate the solution at time zero, so the initial condition makes sense, and the solution indeed satisfies it: $u(0) = u_0$.

This theorem is standard (see, e.g., [33, p. 382], [19, §7.2]). With this framework, we can show that a wide class of PDE problems, particularly ones that are suited to finite element approximations, are well-posed.

3.2. The Hodge Heat Equation as an Abstract Evolution Equation

Let us now see how these results apply in the case of the Hodge heat equation (1.1) on manifolds. We take a different approach from what is done in [22] and [1], solving an equivalent problem. This sets things up for our modified numerical method detailed in later sections.

Let (W, d) be a closed Hilbert complex, with domain complex (V, d), the standard setup in the above—in particular, we have the Poincaré inequality and the well-posedness of the continuous Hodge Laplacian problem. We consider the space $\mathfrak{Y}^k := V^{k-1} \times V^k$ and its dual $(\mathfrak{Y}^k)' = (V^{k-1})' \times (V^k)'$ with the obvious product norms (we use primes to denote dual spaces so as not to conflict with the dual complex with respect to the Hodge star defined earlier, though these uses are related). This, along with $H = W^{k-1} \times W^k$, gives rigged Hilbert space structure

$$\mathfrak{Y} \subseteq H \subseteq \mathfrak{Y}'.$$

The embeddings are dense and continuous by definition of the graph inner product and the operators d having dense domain. We consider the **Bochner mixed weak parabolic problem**: to seek a weak solution $(u,\sigma) \in L^2(I,\mathfrak{Y}) \cap H^1(I,\mathfrak{Y}')$ to the mixed problem

$$\begin{aligned} &\langle \sigma, \omega\rangle - \langle u, d\omega\rangle = 0, && \forall\, \omega \in V^{k-1}, \quad t \in I, \\ &\langle u_t, \varphi\rangle + \langle du, d\varphi\rangle + \langle d\sigma, \varphi\rangle = \langle f, \varphi\rangle, && \forall\, \varphi \in V^k, \quad t \in I, \\ &u(0) = g. \end{aligned} \tag{3.5}$$

We see that (3.5) is the mixed form of (1.1), as in the elliptic case, introducing the variable σ defined by $\sigma = d^*u$, where d^* is the adjoint of the operator d. We write the equation weakly (moving d^* back to the other side), which makes no difference at the continuous level, but will make a significant difference when discretizing.

In order to use the abstract results above, we need a term with σ_t. Formally differentiating the first equation of (1.2), and substituting $\varphi = d\omega$ in the second equation, we obtain

$$0 = \langle \sigma_t, \omega\rangle - \langle u_t, d\omega\rangle = \langle \sigma_t, \omega\rangle - \langle f, d\omega\rangle + \langle d\sigma, d\omega\rangle + \langle du, dd\omega\rangle.$$

Since $d^2 = 0$, that last term vanishes, and so we have the following system:

$$\begin{aligned} &\langle \sigma_t, \omega\rangle + \langle d\sigma, d\omega\rangle = \langle f, d\omega\rangle, && \forall\, \omega \in V^{k-1}, \quad t \in I, \\ &\langle u_t, \varphi\rangle + \langle d\sigma, \varphi\rangle + \langle du, d\varphi\rangle = \langle f, \varphi\rangle, && \forall\, \varphi \in V^k, \quad t \in I, \\ &u(0) = g. \end{aligned} \tag{3.6}$$

Theorem 3.2. *Suppose the initial condition g is in the domain of the adjoint V^* and $f \in L^2(I,(V^k)')$. Then the problem* (3.6) *is well-posed: there exists a unique solution $(\sigma,u) \in L^2(I,\mathfrak{Y}) \cap H^1(I,\mathfrak{Y}') \cap C(I,H)$ with $(\sigma(0),u(0)) = (d^*g,g)$.*

Proof. We see that given $f \in L^2(I,(V^k)')$, we have that the functional $F : (\tau,v) \mapsto \langle f, d\tau\rangle + \langle f, v\rangle$ is in $L^2(I,\mathfrak{Y}')$, since d maps V^{k-1} to V^k. For an initial condition on σ, we can demand that $\sigma(0)$ be the unique σ_0 statisfying $\langle \sigma_0, \tau\rangle - \langle g, d\tau\rangle = 0$. For this to reasonably hold, we must actually have at least $u_0 \in V_k^*$, the domain of the adjoint operator d^*, that is, $\sigma_0 = d^*g$. We equip the spaces with the standard inner products for product spaces:

$$\langle (\sigma,u),(\tau,v)\rangle_H := \langle \sigma,\tau\rangle + \langle u,v\rangle \tag{3.7}$$

$$\langle (\sigma,u),(\tau,v)\rangle_{\mathfrak{Y}} := \langle \sigma,\tau\rangle_V + \langle u,v\rangle_V. \tag{3.8}$$

Consider the operator $A : \mathfrak{Y} \to \mathfrak{Y}'$ defined by

$$a(\sigma,u;\omega,\varphi) = \langle -A(\sigma,u),(\omega,\varphi)\rangle = \langle d\sigma, d\omega\rangle + \langle d\sigma,\varphi\rangle + \langle du, d\varphi\rangle.$$

With the functional F defined as above, we have $F \in L^2(I,\mathfrak{Y}')$, and so (3.6) is equivalent to the problem

$$(\sigma,u)_t = A(\sigma,u) + F. \tag{3.9}$$

We now verify that the bilinear form a satisfies Gårding's Inequality:

$$\begin{aligned} a(\sigma, u; \sigma, u) &= \|d\sigma\|^2 + \langle d\sigma, u\rangle + \|du\|^2 \\ &= \|\sigma\|_V^2 - \|\sigma\|^2 + \langle d\sigma, u\rangle + \|u\|_V^2 - \|u\|^2 \\ &\geq \|\sigma\|_V^2 - \|\sigma\|^2 - \|d\sigma\|\,\|u\| + \|u\|_V^2 - \|u\|^2 \\ &\geq \|\sigma\|_V^2 - \|\sigma\|^2 - \frac{1}{2}\|\sigma\|_V^2 - \frac{1}{2}\|u\|_V^2 + \|u\|_V^2 - \|u\|^2 \\ &= \frac{1}{2}\|(\sigma, u)\|_{\mathfrak{Y}}^2 - \|(\sigma, u)\|_H^2. \end{aligned}$$

Thus, the abstract theory applies, and noting that the initial conditions $(d^*g, g) \in H$, we have that

$$(\sigma, u) \in L^2(I, \mathfrak{Y}) \cap H^1(I, \mathfrak{Y}') \cap C(I, H)$$

is the unique solution to problem (3.6) with initial conditions given by $u(0) = g \in V_k^*$ and $\sigma(0) = d^*g$. □

Given this, however, we must still establish that we also have a solution to the original mixed problem (which will be crucial in our error estimates):

Theorem 3.3. *Let $(\sigma, u) \in L^2(I, \mathfrak{Y}) \cap H^1(I, \mathfrak{Y}') \cap C(I, H)$ solve (3.6) above with the initial conditions. Then, in fact, (σ, u) also solves (3.5).*

Proof. The second equation already holds, as it is incorporated unchanged into the equations (3.6). To show the first equation, we show

$$\langle \sigma_t, \omega\rangle - \langle u_t, d\omega\rangle = 0$$

for all time t. Then, since the original mixed equation holds at the initial time, standard uniqueness ensures it holds for all $t \in I$. We simply realize it is setting $\varphi = -d\omega$:

$$\begin{aligned} &\langle \sigma_t, \omega\rangle - \langle u_t, d\omega\rangle \\ =&\langle (\sigma, u)_t, (\omega, -d\omega)\rangle_H = a(\sigma_t, u_t; \omega, -d\omega) + \langle f, d\omega\rangle + \langle f, -d\omega\rangle \\ =&\langle d\sigma, d\omega\rangle + \langle d\sigma, -d\omega\rangle + \langle du, dd\omega\rangle = 0. \end{aligned}$$

This completes the proof. □

4. Error Estimates for the Abstract Parabolic Problem

We now combine all the preceding abstract theory (the Holst-Stern [24] framework recalled in §2.2, and the abstract evolution problems framework recalled in §3) to extend the error estimates of Gillette, Holst, and Zhu [22]. In particular, we recover the case of approximating parabolic equations on compact, oriented[1)] Riemannian hypersurfaces in $\mathbb{R}^{n+1}$ with triangulations in a tubular neighborhood. The key equation in the derivation of the estimates are the generalizations of Thomée's evolution equations for the error terms. We shall see that these equations lead most naturally to the use of certain Bochner norms for the error estimates that are different for each component in the equation.

[1)] Using differential pseudoforms ([20, §2.8], [36]), we can eliminate this restriction. However, more theory needs to be developed for that case; the normal projection, in particular. We consider this in future work.

Let (W, d) be a closed Hilbert complex with domain (V, d), and the Gelfand triple $\mathfrak{Y} \subseteq H \subseteq \mathfrak{Y}'$ on this complex as above. Now consider our previous standard setup of finite-dimensional approximating complexes (W_h, d) with domain (V_h, d), with corresponding spaces $\mathfrak{Y}_h^k = V_h^{k-1} \times V_h^k$ (it is $\mathfrak{X}_h^k$ missing the harmonic factor), $i_h : V_h \hookrightarrow V$ injective morphisms (that are W-bounded), $\pi_h : V_h \to V$ projection morphisms (which may be merely V-bounded), and $\pi_h \circ i_h =$ id. Finally, we consider data interpolation operators $\Pi_h : W \to W_h$, such that $\Pi_h \circ i_h =$ id that realize which projections for the inhomogeneous and prescribed harmonic terms (f_h and w_h in the abstract theory above) that we use.

4.1. Discretization of the weak form

Suppose we have $f \in L^2(I, (V^k)')$ and $g \in V_k^*$. Let $(\sigma, u) \in L^2(I, \mathfrak{Y}) \cap H^1(I, \mathfrak{Y}') \cap C(I, H)$ be the unique (continuous) solution to (3.5), as covered in §3. As in [22], we can consider approximations to this solution as functionals on finite-dimensional spaces $\mathfrak{Y}_h$, e.g. finite element spaces. With the above considerations, we formulate the **semi-discrete Bochner parabolic problem**: Find $(\sigma_h, u_h) : I \to \mathfrak{Y}_h$ such that

$$\begin{aligned} &\langle \sigma_h, \omega_h \rangle_h - \langle u_h, d\omega_h \rangle_h = 0, && \forall\, \omega_h \in V_h^{k-1}, \quad t \in I \\ &\langle u_{h,t}, \varphi_h \rangle_h + \langle d\sigma_h, \varphi_h \rangle_h + \langle du_h, d\varphi_h \rangle_h = \langle \Pi_h f, \varphi_h \rangle_h, && \forall\, \varphi_h \in V_h^k, \quad t \in I \\ &u_h(0) = g_h. \end{aligned} \tag{4.1}$$

(We use the notation of Thomée for the test forms.) We define g_h, the projected initial data, shortly. A similar argument as in §3 above, differentiating the first equation with respect to time, considering the Gelfand triple $\mathfrak{Y}_h^k \subseteq W_h^{k-1} \times W_h^k \subseteq (\mathfrak{Y}_h^k)'$ gives that this problem is well-posed (or more simply, we choose bases and reduce to standard ODE theory as in (1.3) above). Following Gillette, Holst, and Zhu [22], we define the **time-ignorant discrete problem**, using the idea of elliptic projection [37] which we use to define a discrete solution via elliptic projection of the continuous solution at each time $t_0 \in I$: We seek $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h) \in \mathfrak{X}_h^k$ such that

$$\begin{aligned} &\langle \tilde{\sigma}_h, \omega_h \rangle_h - \langle \tilde{u}_h, d\omega_h \rangle_h = 0, && \forall\, \omega_h \in V_h^{k-1} \\ &\langle d\tilde{\sigma}_h, \varphi_h \rangle_h + \langle d\tilde{u}_h, d\varphi_h \rangle_h + \langle \tilde{p}_h, \varphi_h \rangle_h = \langle \Pi_h(-\Delta u(t_0)), \varphi_h \rangle_h, && \forall\, \varphi_h \in V_h^k \\ &\langle \tilde{u}_h, q_h \rangle_h = \langle \Pi_h(P_{\mathfrak{H}} u(t_0)), q_h \rangle_h && \forall\, q_h \in \mathfrak{H}_h^k. \end{aligned} \tag{4.2}$$

Note that we have included a prescribed harmonic form given by the harmonic part of u (following [1]). We then take the initial data g_h to be $\tilde{u}_h(0)$; it is just the solution to the elliptic problem with load data $\Pi_h(-\Delta g)$, since $u(0) = g$. Note we do not directly interpolate g itself via Π_h for the data; the reason for this will be seen shortly. This discrete problem is well-posed, i.e., a unique solution $u_h(t_0)$ always exists for every time $t_0 \in I$, by the first part of Theorem 2.8 above. The presence of an additional term $\tilde{p}_h$ and equation involving harmonic forms departs from Gillette, Holst, and Zhu [22], because the theory there is facilitated by the fact that there are no harmonic n-forms on open domains in $\mathbb{R}^n$ (the *natural* boundary conditions for such spaces are Dirichlet boundary conditions, in contrast to the more classical example of 0-forms, i.e. functions). Here, however, we must consider harmonic forms, since we may not be working at the end of an abstract Hilbert complex. For our model problem, namely differential forms on compact orientable manifolds (without boundary), even in the case of n-forms, the theory

is completely symmetric (by Poincaré duality [5, 28, 31]).[1] In addition, the linear projections Π_h may not preserve the harmonic space, which gives the possibility of a nonzero $\tilde{p}_h$, despite $-\Delta u$ having zero harmonic part (so it is its own error term).

4.2. Determining the error terms and their evolution

Continuing the method of Thomée [35], we use the time-ignorant discrete solution as an intermediate reference, and estimate the total errors by comparing to this reference and using the triangle inequality. Roughly speaking, we try to estimate as follows:

$$\|i_h\sigma_h(t) - \sigma(t)\|_V \le \|i_h\sigma_h(t) - i_h\tilde{\sigma}_h(t)\|_V + \|i_h\tilde{\sigma}_h(t) - \sigma(t)\|_V, \tag{4.3}$$

$$\|i_h u_h(t) - u(t)\|_V \le \|i_h u_h(t) - i_h\tilde{u}_h(t)\|_V + \|i_h\tilde{u}_h(t) - u(t)\|_V. \tag{4.4}$$

It turns out that this grouping of the terms is not the most natural for our purposes. We shall see it is the structure of the error evolution equations that groups the terms more naturally as:

$$\|i_h u_h(t) - u(t)\|, \tag{4.5}$$

$$\|i_h\sigma_h(t) - \sigma(t)\| + \|d(i_h u_h(t) - u(t))\|, \tag{4.6}$$

$$\|d(i_h\sigma_h(t) - \sigma(t))\|. \tag{4.7}$$

The sum of these three terms is the sum of the two V-norms above. In addition, we shall see in our application to hypersurfaces (approximated by curved triangulations of mesh size h) that this particular grouping of the error terms also corresponds more precisely to the order of approximations in the improved estimates for the elliptic projection (namely, they are of orders h^{r+1}, h^r, and h^{r-1}, respectively, for finite element spaces consisting degree-r polynomial differential forms).

The plan is to use the theory of Holst and Stern [24] reviewed in §2.2 above to estimate the sum of the two second terms in (4.3) and (4.4); the elliptic projection simply is an approximation, at each fixed time, of the trivial case of u being the solution of the continuous problem with data given by its own Laplacian, $-\Delta u$. The harmonic form portion will come up naturally as part of the calculuation. Using the notation of Thomée [35], we define the error functions

$$\rho(t) := \tilde{u}_h(t) - i_h^* u(t), \tag{4.8}$$

$$\theta(t) := u_h(t) - \tilde{u}_h(t), \tag{4.9}$$

$$\psi(t) = \tilde{\sigma}_h(t) - i_h^*\sigma(t), \tag{4.10}$$

$$\varepsilon(t) := \sigma_h(t) - \tilde{\sigma}_h(t). \tag{4.11}$$

(Thomée does not define the third term ψ; we have added it for convenience.) In the case that there are no variational crimes (i.e., J_h is unitary), the error terms ρ and ψ are bounded above by the elliptic projection errors (because there, i_h^* is the orthogonal projection, and $\|i_h^*\| = \|i_h\| = 1$), so that we have, for example, that $\|i_h u_h - u\| \le \|\theta\| + \|\rho\|$, corresponding to the use of ρ in [22, 35]. For our purposes, however, the choice of ρ here does not correspond as neatly, now being an intermediate quantity that helps us estimate θ in terms the elliptic projection error (the second term in (4.4)). We find that it contributes more terms with

[1] Despite this, there are a number of reasons why one should still prefer to continue to phrase problems in terms of n-forms if the problem calls for it ([20] describes how it affects the interpretation of certain quantities); and we shall see that it does in fact still make a difference at the discrete level.

$\|I - J_h\|$. Similar remarks apply for σ and ψ. We use the method of Thomée to estimate the terms θ and ε in terms of (the time derivatives of) ρ and ψ, and the elliptic projection error; In order to do this, we need an analogue of Thomée's error equations.

Lemma 4.1 (Generalized Thomée error equations). *Let θ, ρ, and ε be defined as above. Then for all $t \in I$,*

$$\begin{aligned} &\langle \varepsilon, \omega_h \rangle_h - \langle \theta, d\omega_h \rangle_h = 0, && \forall \omega_h \in V_h^{k-1}, \\ &\langle \theta_t, \varphi_h \rangle_h + \langle d\varepsilon, \varphi_h \rangle_h + \langle d\theta, d\varphi_h \rangle_h = \langle -\rho_t + \tilde{p}_h + (\Pi_h - i_h^*) u_t, \varphi_h \rangle_h, && \forall \varphi_h \in V_h^k. \end{aligned} \tag{4.12}$$

This differs from Thomée [35] and Gillette, Holst, and Zhu [22] with the harmonic term $\tilde{p}_h$, which accounts for the projections Π_h possibly not sending the harmonic forms to the discrete harmonic forms, an extra $d\theta$ term which accounts for possibly working away from the end of the complex (for differential forms on an n-manifold, forms of degree $k < n$), and another data interpolation error term for u_t (which also distinguishes it from Arnold and Chen [1]).

Proof. The first equation is simply weakly expressing ε as $d_h^* \theta$. This follows immediately from the corresponding equations in the semidiscrete problem and the time-ignorant discrete problem. For the second term, consider the expression

$$B := \langle \theta_t, \varphi_h \rangle_h + \langle d\varepsilon, \varphi_h \rangle_h + \langle d\theta, d\varphi_h \rangle_h + \langle \rho_t, \varphi_h \rangle_h, \tag{4.13}$$

and expand it using the definitions to obtain

$$\begin{aligned} B = &\langle u_{h,t}, \varphi_h \rangle_h - \langle \tilde{u}_{h,t}, \varphi_h \rangle_h + \langle d\sigma_h - d\tilde{\sigma}_h, \varphi_h \rangle_h \\ &+ \langle du_h - d\tilde{u}_h, d\varphi \rangle_h + \langle \tilde{u}_{h,t}, \varphi_h \rangle_h - \langle i_h^* u_t, \varphi_h \rangle_h. \end{aligned}$$

We cancel the $\tilde{u}_{h,t}$ terms, and apply the semidiscrete equation (4.1) to cancel the $d\sigma_h$ and du_h terms, which gives us

$$B = \langle \Pi_h f, \varphi_h \rangle_h - \langle d\tilde{\sigma}_h, \varphi_h \rangle_h - \langle d\tilde{u}_h, d\varphi_h \rangle_h - \langle i_h^* u_t, \varphi_h \rangle_h,$$

and finally, using the second equation of (4.2) to account for the middle terms, we have

$$\begin{aligned} B =& \langle \Pi_h f, \varphi_h \rangle_h + \langle \Pi_h (\Delta u), \varphi \rangle_h + \langle \tilde{p}_h, \varphi_h \rangle_h - \langle i_h^* u_t, \varphi_h \rangle_h \\ =& \langle \Pi_h \left(\Delta u + f - u_t \right), \varphi_h \rangle_h + \langle \tilde{p}_h, \varphi_h \rangle_h + \langle (\Pi_h - i_h^*) u_t, \varphi_h \rangle_h. \end{aligned}$$

But since $u_t = \Delta u + f$ is the strong form of the equation, which we know is satisfied by the uniqueness, it follows that $B = \langle \tilde{p}_h + (\Pi_h - i_h^*) u_t, \varphi_h \rangle_h$. Subtracting the ρ_t from both sides gives the result. □

Now we present our main theorem.

Theorem 4.1 (Main parabolic error estimates). *Let (σ, u) be the solution to the continuous problem* (3.5)*, (σ_h, u_h) be the semidiscrete solution* (4.1)*, $(\tilde{\sigma}_h, \tilde{u}_h)$ the elliptic projection* (4.2)*, and the error quantities* (4.8)-(4.11) *be defined as above. Then we have the following error estimates:*

$$\|\theta(t)\|_h \le \|\rho_t\|_{L^1(I,W_h)} + \|\tilde{p}_h\|_{L^1(I,W_h)} + \|(\Pi_h - i_h^*) u_t\|_{L^1(I,W_h)}, \tag{4.14}$$

$$\|d\theta(t)\|_h + \|\varepsilon(t)\|_h \le C \left(\|\rho_t\|_{L^2(I,W_h)} + \|\tilde{p}_h\|_{L^2(I,W_h)} + \|(\Pi_h - i_h^*) u_t\|_{L^2(I,W_h)} \right), \tag{4.15}$$

$$\|d\varepsilon(t)\|_h \le C \left(\|\psi_t\|_{L^2(I,W_h)} + \|d_h^* (\Pi_h - i_h^*) u_t\|_{L^2(I,W_h)} \right), \tag{4.16}$$

with

$$\|\rho_t\|_{L^2(I,W_h)} \leq C\left(\|i_h\tilde{u}_{h,t} - u_t\|_{L^2(I,W)} + \|I - J_h\|_{\mathcal{L}(W_h)}\|u_t\|_{L^2(I,W)}\right) \tag{4.17}$$

$$\|\psi_t\|_{L^2(I,W_h)} \leq C\left(\|i_h\tilde{\sigma}_{h,t} - \sigma_t\|_{L^2(I,W)} + \|I - J_h\|_{\mathcal{L}(W_h)}\|\sigma_t\|_{L^2(I,W)}\right). \tag{4.18}$$

We may further combine these terms, which we shall do in a separate corollary, but it is useful to keep things separate; this allows terms to be analyzed individually when considering specific choices of V and V_h. The error terms $i_h\tilde{\sigma}_h - \sigma$ and $i_h\tilde{u}_h - u$ and their time derivatives are furthermore estimated in terms of best approximation norms and variational crimes via the theory of Holst and Stern [24]. The different Bochner norms involved arise from the structure of the error evolution equations.

Proof. We adapt the proof technique in [22, 35] to our situation, and for ease of notation, unsubscripted norms will denote the W-norms and norms subscripted with just h will denote norms on the approximating complex. We now assemble the estimates above separately by computing the W-norms of the errors and their differentials. We begin by estimating $\|\theta(t)\|_h$. We use the standard technique of using the solutions as their own test functions: Set $\varphi_h = \theta$ and $\omega_h = \varepsilon$ in (4.12). Adding the two equations together yields

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_h^2 + \|\varepsilon\|_h^2 + \|d\theta\|_h^2 = \langle -\rho_t + \tilde{p}_h + (\Pi_h - i_h^*)u_t, \theta\rangle_h, \quad t \in I \tag{4.19}$$

Following Thomée [35], we introduce $\delta > 0$ to account for non-differentiability at $\theta = 0$, drop the positive terms $\|\varepsilon\|_h^2 + \|d\theta\|_h^2$, and observe that

$$\begin{aligned}(\|\theta\|_h^2 + \delta^2)^{1/2}\frac{d}{dt}(\|\theta\|_h^2 + \delta^2)^{1/2} &= \frac{1}{2}\frac{d}{dt}(\|\theta\|_h^2 + \delta^2)\\ =\frac{1}{2}\frac{d}{dt}\|\theta\|_h^2 &\leq (\|\rho_t\|_h + \|\tilde{p}_h\|_h + \|(\Pi_h - i_h^*)u_t\|_h)\|\theta\|_h,\end{aligned}$$

using (4.19), the Cauchy-Schwarz inequality, and the definition of operator norms (our goal is to get all of those quantities on the right side of the equation close to zero, so we do not need to care too much about their sign). Thus, since $\|\theta\|_h \leq (\|\theta\|_h^2 + \delta^2)^{1/2}$, we have, canceling $\|\theta\|_h$,

$$\frac{d}{dt}(\|\theta\|_h^2 + \delta^2)^{1/2} \leq \|\rho_t\|_h + \|\tilde{p}_h\|_h + \|(\Pi_h - i_h^*)u_t\|_h.$$

Now, using the Fundamental Theorem of Calculus, we integrate from 0 to t to get

$$\begin{aligned}\|\theta(t)\|_h =& \|\theta(0)\|_h + \lim_{\delta\to 0}\int_0^t \frac{d}{dt}(\|\theta\|_h^2 + \delta^2)^{1/2}\\ &\leq \int_0^t (\|\rho_t\|_h + \|\tilde{p}_h\|_h + \|(\Pi_h - i_h^*)u_t\|_h).\end{aligned} \tag{4.20}$$

$\theta(0)$ vanishes by our choice of initial condition as the elliptic projection.

Next, continuing to follow [22], we consider $\|\varepsilon(t)\|_h$. We differentiate the first error equation and substitute $\varphi_h = 2\theta_t$ and $\omega_h = 2\varepsilon$, so that

$$\langle \varepsilon_t, 2\varepsilon\rangle_h - \langle \theta_t, 2d\varepsilon\rangle_h = 0, \tag{4.21}$$

$$\langle \theta_t, 2\theta_t\rangle_h + \langle d\varepsilon, 2\theta_t\rangle_h + \langle d\theta, 2d\theta_t\rangle_h = \langle -\rho_t + \tilde{p}_h + (\Pi_h - i_h^*)u_t, 2\theta_t\rangle_h. \tag{4.22}$$

Adding the two equations as before, we have, by Cauchy-Schwarz and the AM-GM inequality (here in the form $2ab \le \frac{3}{2}a^2 + \frac{2}{3}b^2$),

$$\begin{aligned}
&\frac{d}{dt}\|\varepsilon\|_h^2 + 2\|\theta_t\|_h^2 + \frac{d}{dt}\|d\theta\|_h^2 \\
\le& 2\|\rho_t\|_h\|\theta_t\|_h + 2\|\tilde{p}_h\|_h\|\theta_t\|_h + 2\|(\Pi_h - i_h^*)u_t\|_h\|\theta_t\|_h \\
\le& \tfrac{3}{2}\left(\|\rho_t\|_h^2 + \|\tilde{p}_h\|_h^2 + \|(\Pi_h - i_h^*)u_t\|_h^2\right) + 2\|\theta_t\|_h^2.
\end{aligned}$$

Canceling $2\|\theta_t\|_h^2$ from both sides, using the Fundamental Theorem of Calculus, and noting the initial conditions vanish by the choice of elliptic projection, we have

$$\|\varepsilon\|_h^2 + \|d\theta\|_h^2 \le \tfrac{3}{2}\int_0^t \left(\|\rho_t\|_h^2 + \|\tilde{p}_h\|_h^2 + \|(\Pi_h - i_h^*)u_t\|_h^2\right). \tag{4.23}$$

Finally, we estimate $\|d\varepsilon\|_h$. As in the estimate above, we differentiate the first equation with respect to time, and substitute $\omega = 2\varepsilon_t$, $\varphi = 2d\varepsilon_t$,

$$\langle \varepsilon_t, 2\varepsilon_t\rangle_h - \langle \theta_t, 2d\varepsilon_t\rangle_h = 0 \tag{4.24}$$

$$\langle \theta_t, 2d\varepsilon_t\rangle_h + \langle d\varepsilon, 2d\varepsilon_t\rangle_h + \langle d\theta, 2dd\varepsilon_t\rangle_h = \langle -\rho_t + \tilde{p}_h + (\Pi_h - i_h^*)u_t, 2d\varepsilon_t\rangle_h. \tag{4.25}$$

Noting that $d^2 = 0$, $\tilde{p}_h$ is perpendicular to the coboundaries, and $\psi = d_h^*\rho$, we add the equations to get

$$\begin{aligned}
&2\|\varepsilon_t\|_h^2 + \frac{d}{dt}\|d\varepsilon\|_h^2 = 2\langle -\rho_t + (\Pi_h - i_h^*)u_t, d\varepsilon_t\rangle_h \\
=& 2\langle -\psi_t + d_h^*(\Pi_h - i_h^*)u_t, \varepsilon_t\rangle_h \le \|\psi_t\|_h^2 + \|d_h^*(\Pi_h - i_h^*)u_t\|_h^2 + 2\|\varepsilon_t\|_h^2.
\end{aligned}$$

By the Fundamental Theorem of Calculus, and noting vanishing initial conditions (and an exact cancellation of positive terms), we have

$$\|d\varepsilon\|_h^2 \le \int_0^t \left(\|\psi_t\|_h^2 + \|d_h^*(\Pi_h - i_h^*)u_t\|_h^2\right). \tag{4.26}$$

We now estimate ρ and ψ. We note that the time derivatives of the solutions are also solutions to the mixed formulation, at least provided that u_t and other associated quantities are sufficiently regular (in the domain of the Laplace operator) for the norms and derivatives to make sense. Then (recalling $i_h^+ = J_h^{-1} i_h^*$), we have

$$\begin{aligned}
\|\rho_t(t)\|_h &= \|\tilde{u}_{h,t} - i_h^* u_t\| \le \|\tilde{u}_{h,t} - i_h^+ u_t\| + \|i_h^+ u_t - i_h^* u_t\| \\
&\le \|i_h^+\|\Big(\|i_h\tilde{u}_{h,t} - u_t\| + \|I - J_h\|\|u_t\|\Big), \qquad (4.27)\\
\|\psi_t(t)\|_h &= \|\tilde{\sigma}_{h,t} - i_h^*\sigma_t\| \le \|\tilde{\sigma}_{h,t} - i_h^+\sigma_t\| + \|i_h^+\sigma_t - i_h^*\sigma_t\| \\
&\le \|i_h^+\|\Big(\|i_h\tilde{\sigma}_{h,t} - \sigma_t\| + \|I - J_h\|\,\|\sigma_t\|\Big). \qquad (4.28)
\end{aligned}$$

We note that the two first terms in the last estimates above actually can be further expressed in terms of best approximation errors, but involving time derivatives of the solution, using the theory of §2.2. The abstract theory actually uses V-norms, which trivially can be used to overestimate the W-norms here. We cannot improve this in the abstract theory; instead, we use theory for specific choices of V, W, and V_h, such as appropriately chosen de Rham complexes

and approximations to improve the estimates ([3, §3.5], [1, Theorem 3.1]). For these cases, it is helpful to keep the individual estimates on $\|\varepsilon\|^2$, $\|\theta\|^2$, etc. separated. We have combined terms because the abstract theory gives us all the variational crimes together, as it makes heavy use of the bilinear forms above. Additional improvement of estimates based on regularity as done in [3] cannot made for the variational crimes, as discussed in [24, §3.4]. We give the relevant example and result in the next section.

Corollary 4.1 (Combined L^1 estimate). *Let θ, ρ, ψ, and ε be as above. Then we have*

$$\begin{aligned}&\|i_h\sigma_h-\sigma\|_{L^1(I,V)}+\|i_hu_h-u\|_{L^1(I,V)}\\ \le& C\Big(\|\rho_t\|_{L^2(I,W_h)}+\|(\Pi_h-i_h^*)u_t\|_{L^2(I,W_h)}+\|\psi_t\|_{L^2(I,W_h)}+\|d_h^*(\Pi_h-i_h^*)u_t\|_{L^2(I,W_h)}\\ &\|\rho_t\|_{L^2(I,W_h)}+\|\tilde{p}_h\|_{L^2(I,W_h)}+\|i_h\tilde{\sigma}_h-\sigma\|_{L^2(I,V)}+\|i_h\tilde{u}_h-u\|_{L^2(I,V)}\Big).\end{aligned}\tag{4.29}$$

Further expanding the time derivative terms, we have

$$\begin{aligned}&\|i_h\sigma_h-\sigma\|_{L^1(I,V)}+\|i_hu_h-u\|_{L^1(I,V)}\\ \le& C\Big(\|i_h\tilde{u}_{h,t}-u_t\|_{L^2(I,W)}+\|i_h\tilde{\sigma}_{h,t}-\sigma_t\|_{L^2(I,W)}+\|I-J_h\|\,\|u_t\|_{L^2(I,W)}\\ &+\|I-J_h\|\,\|\sigma_t\|_{L^2(I,W)}+\|(\Pi_h-i_h^*)u_t\|_{L^2(I,W_h)}+\|d_h^*(\Pi_h-i_h^*)u_t\|_{L^2(I,W_h)}\\ &+\|i_h\tilde{p}_h\|_{L^2(I,W)}+\|i_h\tilde{\sigma}_h-\sigma\|_{L^2(I,V)}+\|i_h\tilde{u}_h-u\|_{L^2(I,V)}\Big).\end{aligned}$$

These terms are organized as follows: the W-error in the approximations of the time derivatives, the variational crimes with $\|I-J_h\|$, the data approximation error for the time derivatives, and finally the V-approximation errors for the elliptic projection. These can be further expanded in terms of best approximation errors, but we will not have use for that outside of specific examples where the computation is easier done with the previous theorems. This corollary is simply stated for conceptual clarity and a qualitative sense of all the different individual contributions to the error.

Proof. First, expanding out $\|i_h\sigma_h-\sigma\|_V+\|i_hu_h-u\|_V$ in terms of W-norms, using the triangle inequality with $i_h\tilde{u}_h$ and $i_h\tilde{\sigma}_h$, and taking out a factor of the norm $\|i_h\|$, we have

$$\begin{aligned}&\|i_h\sigma_h-\sigma\|_V+\|i_hu_h-u\|_V\\ \le&\|i_h\|\Big(\|\theta\|_h+\|d\theta\|_h+\|\varepsilon\|_h+\|d\varepsilon\|_h+\|i_h\tilde{u}_h-u\|_V+\|i_h\tilde{\sigma}_h-\sigma\|_V\Big),\end{aligned}$$

and we recognize the sum of all three left sides of (4.14)–(4.16). We note that by the Cauchy-Schwarz inequality, the estimate for $\|d\theta(t)\|+\|\varepsilon(t)\|$ in (4.15) and $\|d\varepsilon(t)\|$ in (4.16) can be rewritten as using $L^2(I,W)$ norms to match the squared terms (4.23) and (4.26), respectively. Using the right sides of (4.14)–(4.16), and combining and absorbing constants, we arrive at

$$\begin{aligned}&\|i_h\sigma_h-\sigma\|_V+\|i_hu_h-u\|_V\\ \le& C\Big(\|\rho_t\|_{L^2(I,W_h)}+\|(\Pi_h-i_h^*)u_t\|_{L^2(I,W_h)}+\|\psi_t\|_{L^2(I,W_h)}+\|d_h^*(\Pi_h-i_h^*)u_t\|_{L^2(I,W_h)}\\ &+\|\tilde{p}_h\|_{L^2(I,W_h)}\Big)+\|i_h\tilde{\sigma}_h-\sigma\|_V+\|i_h\tilde{u}_h-u\|_V.\end{aligned}$$

Integrating from 0 to T, the latter two V-norm terms become $L^1(I,V)$ norms (and the additional factor of T from integrating the rest of the terms can be absorbed into the constant). Finally, using Cauchy-Schwarz to change the $L^1(I,V)$ norm into an $L^2(I,V)$ norm, and substituting for ρ_t and ψ_t gives the result. □

5. Parabolic Equations on Compact Manifolds

As an application of the preceding results, we return to our original motivating example of de Rham complex to explore an example with the Hodge heat equation on hypersurfaces of Euclidean space, generalizing the discussion in [22, 24]. Let M be compact hypersurface embedded in $\mathbb{R}^{n+1}$. M inherits a Riemannian metric from the Euclidean metric of $\mathbb{R}^{n+1}$.

5.1. The de Rham Complex on a Manifold

We define the L^2 differential k-forms on M given by

$$L^2\Omega^k(M) := \left\{ \sum_{1\le i_1<\cdots<i_k\le n} a_{i_1\ldots i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k} \in \Omega^k(M) \ : \ a_{i_1\ldots i_k}\in L^2(M) \right\},$$

the standard indexing of differential form basis elements, namely strictly increasing sequences from $\{1,\ldots,n\}$. The inner product is given by $\langle\omega,\eta\rangle = \int \omega\wedge\star\eta$, where $\star$ is the Hodge operator corresponding to the metric.

The weak exterior derivative d^k is defined on the domains $H\Omega^k(M)$, and we have a Hilbert complex $(L^2\Omega, d)$ with domain complex $(H\Omega(M), d)$, with $d^{k+1}\circ d^k = 0$:

$$0 \longrightarrow H\Omega^0 \xrightarrow{d^0} H\Omega^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H\Omega^n \longrightarrow 0.$$

As required in the abstract Hilbert complex theory, each domain space carries the graph inner product:

$$\langle u,v\rangle_{H\Omega^k(M)} := \langle u,v\rangle_{L^2\Omega^k(M)} + \langle d^k u, d^k v\rangle_{L^2\Omega^{k+1}(M)}.$$

For open subsets $U\subseteq\mathbb{R}^3$, this reduces to familiar Sobolev spaces of vector fields with the traditional gradient, curl, and divergence operators of vector analysis:

$$0 \longrightarrow H^1(U) \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2(U) \longrightarrow 0.$$

Similarly, the dual complex is $H^*\Omega(M)$ defined by $H^*\Omega^k(M) := \star H\Omega^{n-k}(M)$, consisting of Hodge duals of $(n-k)$-forms. We have that the embedding $H\Omega^k(M)\cap H^*\Omega^k(M)\hookrightarrow \mathcal{L}^2\Omega^k(M)$ is compact, which enables a Poincaré Inequality to hold and the resulting Hilbert complex $(L^2\Omega^k(M), d)$ to be a closed complex [3, 32]. To summarize, we have the following:

Theorem 5.1. *Let M be a compact smooth Riemannian hypersurface in $\mathbb{R}^{n+1}$. Then taking $W^k = L^2\Omega^k(M)$, with maps d^k the exterior derivative defined on the domains $V^k = H\Omega^k(M)$, (W,d) is a closed Hilbert complex with domain (V,d).*

We thus are able to define Hodge Laplacians, and see all the abstract theory for the continuous problems (2.15) and (3.6) applies with these choices of spaces.

5.2. Approximation of a hypersurface in a tubular neighborhood

In order to approximate the problems (2.15) and (3.6), we consider, following [24], a family of approximating hypersurfaces M_h to an oriented hypersurface M, without boundary, all contained in a tubular neighborhood U of M. The surfaces M_h generally will be piecewise polynomial (say, of degree s); they may have corners, but no breaks (C^0 regularity). The case

$s = 1$ (piecewise linear functions) corresponds to triangulations, studied in [16, 17]. The case $s > 1$ was later explored in [15, §2.3]. However, the piecewise linear case still is instrumental in the analysis and indeed, the definition of the spaces (via Lagrange interpolation), and so we shall denote it by T_h (the triangulation, i.e., set of simplices, will be correspondingly denoted by $\mathcal{T}_h$, and their images under the interpolation will be denoted $\hat{\mathcal{T}}_h$). It is convenient, also, to assume that the vertices of the both the triangulation and the higher-degree interpolated surfaces actually lie on the true hypersurface.

The normal vector ν to M allows us to define a signed distance function $\mathrm{dist} : U \to \mathbb{R}$ given by

$$\mathrm{dist}(x) = \pm\,\mathrm{dist}(x, M) = \pm \inf_{y \in M} |x - y|$$

where the sign is chosen in accordance to which side of the normal x lies on. By standard analytic results [21, Ch.14], dist is smooth, provided U is small enough; the maximum distance for which it exists is controlled by the sectional curvature of M. The normal ν can be extended to the whole neighborhood; in fact it is the gradient $\nabla\delta$. It is also convenient to define the normals ν_h to the approximating surfaces M_h. We often compare ν_h to the extended ν. See Figure 5.1. In most of the examples we consider, we assume the vertices of M_h (and T_h) lie on M, but this is not a strict requirement. Instead, we need a condition to ensure that the hypersurfaces M_h are homeomorphic to M, eliminating the possibility of a double covering (e.g., as pictured in [18, Fig. 1, p. 12]). In particular, we want M_h to have the same topology as M. This is again restriction on the size of the tubular neighborhood. In such a neighborhood U, every $x \in U$ decomposes *uniquely* as

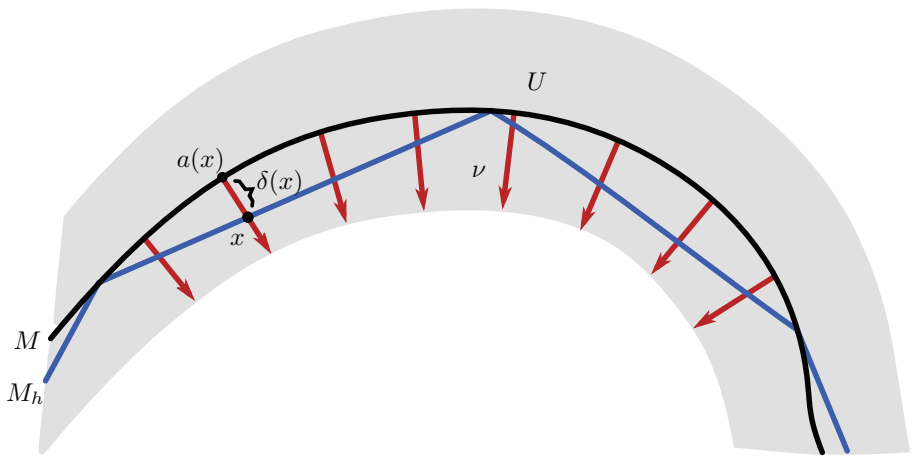

Fig. 5.1. A curve M with a triangulation (blue polygonal curve M_h) within a tubular neighborhood U of M. Some normal vectors ν are drawn, in red; the distance function δ is measured along this normal. The intersection x of the normal with M_h defines a mapping a from x to its base point $a(x) \in M$.

$$x = a(x) + \mathrm{dist}(x)\nu(x), \tag{5.1}$$

where $a(x) \in M$, and $a : U \to M$ is in fact a smooth function, called the **normal projection**. a can then be used to define the degree-s Lagrange interpolated hypersurfaces by considering the image of T_h under the degree-s Lagrange interpolation of a over each simplex in $\mathcal{T}_h$ (we write $a_k : T_h \to M_h$ for this) [15, §2.3].

Example 5.1 (Normal Projections and the Unit Circle). We consider a crude linear mesh interpolation of the unit circle, namely an inscribed square. The mesh T_h is the union of four diagonal line segments. We would like to get a feeling for what the normal projection is and how we would interpolate to find higher-degree approximations. We first consider the upper right quadrant (which is pictured in black in Figure 5.2). We choose x as a coordinate for the segment (pictured in blue in Figure 5.2), and in $\mathbb{R}^2$ it is $(x, 1-x)$ for $x \in [0,1]$. Then the normal projection to the circle is given by the radial line projection

$$a(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

Expressed solely in terms of the coordinate x, we have

$$a(x) = \left(\frac{x}{\sqrt{x^2+(1-x)^2}}, \frac{1-x}{\sqrt{x^2+(1-x)^2}} \right).$$

It is this function that we seek to interpolate (the mapping $x \mapsto (x, 1-x)$ is affine, so concepts like being a polynomial interpolation are still meaningful). Let us now consider a quadratic interpolation over the mesh of a. Considering the nodes 0, $\frac{1}{2}$, and 1, the standard quadratic basis is given by $(\phi_1, \phi_2, \phi_3) = (2x^2 - 3x + 1, 4x - 4x^2, 2x^2 - x)$, and evaluation of a at each node gives us

$$\begin{aligned} a_2(x) &= \left(\frac{1}{\sqrt{2}}\phi_2(x) + \phi_3(x), \phi_1(x) + \frac{1}{\sqrt{2}}\phi_2(x) \right) \\ &= \left((2-2\sqrt{2})x^2 + (2\sqrt{2}-1)x, (2-2\sqrt{2})x^2 + (2\sqrt{2}-3)x + 1 \right). \end{aligned}$$

Note that though a circle is the level set of a quadratic function in x and y, in terms of this specific normal projection, it is definitely not a quadratic function of x alone, so the

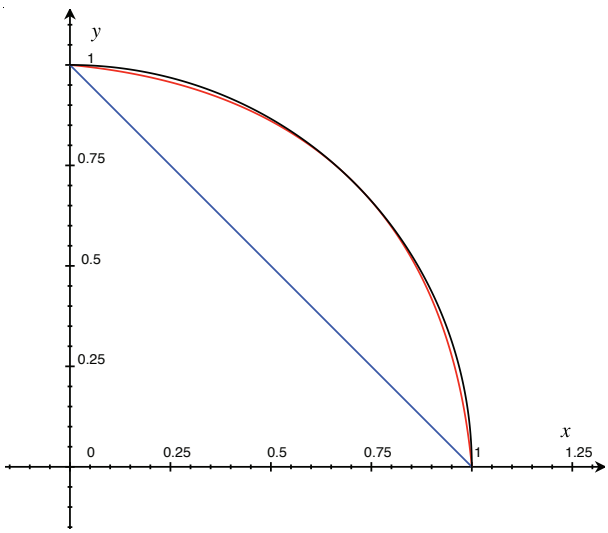

Fig. 5.2. Approximation of a portion of the unit circle (black) with a segment (blue) and much better quadratic Lagrange interpolation for the normal projection (red), though the underlying triangulation is the same (and thus also the mesh size).

quadratic curve will not actually coincide with the circle (although it is close). See Figure 5.2 for a portion of this; the remaining portions of the triangulation and interpolation are all symmetrically reflected across the axes (the quadratic approximation, in red, is so good that the difference would be difficult to see were all parts included).

Restricting a to the surfaces M_h gives diffeomorphisms

$$a|_{M_h} : M_h \to M.$$

$a : M_h \to M$ is therefore a diffeomorphism when restricted to each polyhedron (and is at least globally Lipschitz continuous, the maximum degree of regularity in the piecewise linear case. This is not a problem for Hodge theory, because the form spaces are at most H^1 where regularity is concerned; see [38, Ch.4]).

5.3. Finite element spaces

We choose finite-dimensional subspaces Λ_h^k of $H\Omega^k(M_h)$ for each k, satisfying the subcomplex property $d_h\Lambda_h^k \subseteq \Lambda_h^{k+1}$. We can then pull forms on M_h back to forms on M via the inverse of the normal projection, which furnishes the injective morphisms $i_h^k : \Lambda_h^k \hookrightarrow H\Omega^k(M)$ (since pullbacks commute with d) required by the theory above in Section 2. The main finite element spaces relevant for our purposes are two families of piecewise polynomials, discussed in detail in [2,3]. We must choose these spaces for our equations in a specific relationship in order for the numerical methods and theory detailed above to apply, and for the approximations to work. This is why we prefer a piecewise polynomial approximation of M as opposed to a curved triangulation of M itself; these are shown to have these necessary properties.

These spaces take some work to build; an overview of the process is as follows: first we define the polynomial spaces in $\mathbb{R}^n$, then restrict them to simplices, and then define spaces over the piecewise linear manifold T_h given by a triangulation $\mathcal{T}_h$ of M (with continuity conditions on the boundary simplices). Finally, for higher-level interpolations M_h over T_h, we pull forms on T_h back to M_h via the interpolated normal projections.

Definition 5.1 (Polynomial differential forms). *Let $\mathcal{P}_r$ be polynomials of degree at most r, in n variables (i.e., on $\mathbb{R}^n$), and $\mathcal{H}_r$ be the subspace of homogeneous polynomials of degree r in $\mathbb{R}^n$. We define the first family, denoted $\mathcal{P}_r\Lambda^k$, to consist of all k-forms with coefficients belonging to $\mathcal{P}_r$. The second family, denoted $\mathcal{P}_r^-\Lambda^k$, lie between the spaces of the first class:*

$$\mathcal{P}_{r-1}\Lambda^k \subseteq \mathcal{P}_r^-\Lambda^k \subseteq \mathcal{P}_r\Lambda^k.$$

These are defined as follows: first, consider the radial vector field $X = x^i \frac{\partial}{\partial x^i}$, that is, at each x, it is a radially pointing vector of length $|x|$, and then define the Koszul operator $\kappa\omega := X \lrcorner \omega$, the interior product with X. Then

$$\mathcal{P}_r^-\Lambda^k := \mathcal{P}_{r-1}\Lambda^k \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}.$$

Since κ always raises polynomial degree and decreases form degree, it sends $\mathcal{H}_{r-1}\Lambda^{k+1}$ to $H_r\Lambda^k$, showing that the sum of spaces above is in fact direct. By properties of interior products, $\kappa^2 = 0$.

Definition 5.2 (Polynomial Differential Forms on Simplices). *Let f be a simplex in $\mathbb{R}^n$ of dimension $\ell \leq n$. Then we define $\mathcal{P}_r\Lambda^k(f)$ to be the pullback $i^*\mathcal{P}_r\Lambda^k$ where $i : f \hookrightarrow \mathbb{R}^n$ is*

the inclusion. Two special cases warrant mention. First, if $\ell = n$, then this is just pointwise restriction of polynomial forms in $\mathbb{R}^n$ to the subset f. Second, if $\ell < k$, that is, the simplex dimension is less than the form degree, $\mathcal{P}_r\Lambda^k(f) = 0$. Similarly, $\mathcal{P}_r^-\Lambda^k(f)$ is defined to be $i^*\mathcal{P}_r^-\Lambda^k$. Finally, the space of all forms $\omega \in \mathcal{P}_r\Lambda^k(f)$ such that $\mathrm{Tr}_{f,\partial f}(\omega) = 0$ (forms that vanish in the trace sense on the boundary ∂f) is denoted $\mathring{\mathcal{P}}_r\Lambda^k(f)$. A similar definition holds for the $\mathcal{P}_r^-$ spaces.

It is useful to note that the polynomial spaces $\mathcal{P}_r^-\Lambda^k$ are *affine-invariant*, that is, given an affine map $\phi : \mathbb{R}^n \to \mathbb{R}^n$, $\phi^*\mathcal{P}_r^-\Lambda^k = \mathcal{P}_r^-\Lambda^k$. This ensures that whatever affine parametrizations we choose for simplices f (of any dimension) in $\mathbb{R}^n$, the polynomial space $\mathcal{P}_r^-\Lambda^k(f)$ is the same [3, §5.1.3] (the space $\mathcal{P}_r\Lambda^k$ is also obviously affine-invariant, but it is not so obvious for $\mathcal{P}_r^-$ spaces, because the space $\mathcal{H}_r\Lambda^k$ is *not* affine-invariant).

There are a number of interesting theorems about polynomial form spaces on simplices as described in [3, §5.2], [2, §4.6-4.7], in particular, about the structure of the spaces corresponding to the degrees of freedom, the most interesting of which we paraphrase here:

Theorem 5.2 (Geometric Decomposition Theorem, [3], Theorem 5.5). *Let r, n, and k satisfy $0 \le k \le n$ and $r > 0$, and suppose f is an n-simplex. For any face g of dimension ℓ in f, $\mathcal{P}_{r+k-\ell}^-\Lambda^{\ell-k}(g)$ acts on $\mathcal{P}_r\Lambda^k(f)$ as follows: we associate to each $\eta \in \mathcal{P}_{r+k-\ell}^-\Lambda^{\ell-k}(g)$ the linear mapping*

$$\omega \mapsto \int_g \mathrm{Tr}_{f,g}\,\omega \wedge \eta.$$

Then

$$\mathcal{P}_r\Lambda^k(f)^* \cong \bigoplus_{g \text{ a face}} \mathcal{P}_{r+k-\dim g}^-\Lambda^{\dim g-k}(g)$$

with this correspondence. Similarly,

$$\mathcal{P}_r^-\Lambda^k(f)^* \cong \bigoplus_{g \text{ a face}} \mathcal{P}_{r+k-\dim g-1}\Lambda^{\dim g-k}(g).$$

We refer these mappings as the ***degrees of freedom*** *for the space $\mathcal{P}_r\Lambda^k(f)$ or $\mathcal{P}_r^-\Lambda^k(f)$.*

This simply says that as g varies over all faces $g \subseteq f$, the actions above (and linear combinations thereof) exhaust the entire dual space. It is in this sense that the $\mathcal{P}$ and $\mathcal{P}^-$ spaces are dual to one another. The duality is even more apparent when we consider $\mathring{\mathcal{P}}$ and $\mathring{\mathcal{P}}^-$, because ∂f consists precisely of the simplices of f that are of strictly lower dimension, so the action of any lower dimensional face vanishes, and we have

Corollary 5.1 ([3], Lemma 5.6).

$$\mathring{\mathcal{P}}_r\Lambda^k(f)^* \cong \mathcal{P}_{r+k-n}^-\Lambda^{n-k}(f)$$

and

$$\mathring{\mathcal{P}}_r^-\Lambda^k(f)^* \cong \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(f).$$

This kind of duality is reminiscent of forms versus forms of compact support in, e.g., Poincaré duality. We can now define these spaces over whole triangulations:

Definition 5.3 (Polynomial Finite Element Spaces). *Let $\mathcal{T}$ be a triangulation and T be the piecewise linear manifold $\bigcup \mathcal{T}$. We define the polynomial finite element space $\mathcal{P}_r\Lambda^k(\mathcal{T})$ to be the set of all $\omega \in L^2\Omega^k(T)$ such that for any simplex $f \in \mathcal{T}$, $\omega|_f \in \mathcal{P}_r\Lambda^k(f)$, and for any two simplices f_1, f_2 sharing a common face g, the degrees of freedom yield the same values on $\mathrm{Tr}_{f_1,g}(\omega|_{f_1})$ and $\mathrm{Tr}_{f_2,g}(\omega|_{f_2})$. Equivalently, noting Theorem 5.2 above, they satisfy*

$$\int_g \mathrm{Tr}_{f_1,g}(\omega|_{f_1}) \wedge \eta = \int_g \mathrm{Tr}_{f_2,g}(\omega|_{f_2}) \wedge \eta$$

for all common faces g and $\eta \in \mathcal{P}_{r+k-\dim g}\Lambda^{\dim g-k}(g)$.

These polynomial spaces generalize existing finite element spaces, such as Whitney forms, Nédélec elements, and Raviart-Thomas elements (see [3, 22] for these examples and more). We also have the following important characterization of these spaces:

Theorem 5.3 ([3], Theorem 5.7). *$\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ are $H\Omega^k(T)$ forms that when restricted to any $f \in \mathcal{T}$ are in $\mathcal{P}_r\Lambda^k(f)$ or, respectively, $\mathcal{P}_r^-\Lambda^k(f)$.*

This means, in particular, that the continuity conditions in the definitions above are strong enough that the forms always regular enough to be in the domain complex $(H\Omega, d)$. This theorem holds for triangulations of manifolds (non-flat triangulations) because the corners in the piecewise linear manifold T only occur at the boundaries of simplices, which are compatible with the regularity requirements of our finite element spaces: the regularity of the manifold is C^0, while the finite element spaces are at most H^1.

When formulating the whole discrete cochain complex consisting of spaces Λ_h^k, we recall we must satisfy, at each step, $d\Lambda_h^{k-1} \subseteq \Lambda_h^k$. Other than this restriction, there is considerable choice in choosing a whole chain complex: each target space can possibly be made larger, to become the domain of the next operator. For polynomial spaces, since d decreases polynomial degree and increases form degree, it follows that if $\Lambda_h^k = \mathcal{P}_r\Lambda^k$, one can choose $\Lambda_h^{k+1} \supseteq \mathcal{P}_{r-1}\Lambda^{k+1}$. Following [3, §5.1.4], we will choose either

$$\Lambda_h^{k+1} = \mathcal{P}_{r-1}\Lambda^{k+1}$$

or, for a larger space, the other kind of polynomial space we have studied,

$$\Lambda_h^{k+1} = \mathcal{P}_r^-\Lambda^{k+1}.$$

Of course, to continue the chain, we must also specify what happens if $\Lambda_h^k = \mathcal{P}_r^-\Lambda^k$; for this, we make the same choice $\mathcal{P}_{r-1}\Lambda^k$ or $\mathcal{P}_r^-\Lambda^k$. The general rule is if the next one is a $\mathcal{P}^-$ space, it keeps the same polynomial degree (even though the image of the space will surely have lower degree), and if it is just a $\mathcal{P}$ space, one reduces the polynomial degree, and this holds regardless of whether the current space is a $\mathcal{P}$ or $\mathcal{P}^-$ space. With this choices made for every space in the complex, we have 2^{n-1} possibilities (from 0 to n, there are 2^{n+1} choices, but it is always the case that $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$, while $\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n$, eliminating two places to choose from). Similar considerations hold for choosing complexes on simplices and triangulations (where one choice of $\mathcal{P}^-$ or $\mathcal{P}$ is made to hold over every simplex). Of course, for one single Hodge Laplacian problem, we only need to work with three spaces in the chain, since the equations only involve $(k-1)$- and k-forms and their differentials.

The important property of these spaces is that they admit the cochain projections whose role we have seen is so important in the theory. First, we describe the case as in [3] where $M = U$ is

a bounded polyhedral domain in $\mathbb{R}^n$, and a triangulation $\mathcal{T}$ conforming to the boundary. Then we consider the cochain projections defined in [3, §5.5]:

$$\pi_h^k : L^2\Omega^k \to \Lambda_h^k \quad \text{where } \Lambda_h^k \in \{\mathcal{P}_r\Lambda^k(\mathcal{T}),\ \mathcal{P}_r^-\Lambda^k(\mathcal{T})\}. \tag{5.2}$$

These operators are by construction uniformly bounded (in $L^2\Omega^k$, not just $H\Omega^k$) with respect to h. Finally, the following theorem explicitly expresses the projection error (and hence, best approximation error) in terms of powers of the mesh size h and the norms of the solution.

Theorem 5.4 ([3], Theorem 5.9). (i) *Let Λ_h^k be one of the spaces $\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ or, if $r \geq 1$, $\mathcal{P}_r\Lambda^k(\mathcal{T})$. Then π_h^k is a cochain projection onto Λ_h^k and satisfies*

$$\|\omega - \pi_h^k\omega\|_{L^2\Omega^k(U)} \leq ch^s\|\omega\|_{H^s\Omega^k(U)}, \qquad \omega \in H^s\Omega^k(U),$$

for $0 \leq s \leq r+1$. Moreover, for all $\omega \in L^2\Omega^k(U)$, $\pi_h^k\omega \to \omega$ in L^2 as $h \to 0$.

(ii) *Let Λ_h^k be one of the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ with $r \geq 1$. Then*

$$\|d(\omega - \pi_h^k\omega)\|_{L^2\Omega^k(U)} \leq ch^s\|d\omega\|_{H^s\Omega^k(U)}, \qquad \omega \in H^s\Omega^k(U),$$

for $0 \leq s \leq r$.

These bounded cochain operators are explicitly constructed in [2, 3]. They are constructed by first taking the natural interpolation operators I_h defined for continuous differential forms [3, p. 335], but combined with smoothings to allow extension to H^s differential forms which may not necessarily be continuous [3, §5.5].

We now define the polynomial finite element spaces for degree-s Lagrange interpolated surfaces over the triangulation $\mathcal{T}_h$, where $s > 1$. These spaces generalize the function spaces considered by Demlow [15, §2.5].

Definition 5.4. *Given a triangulation $\mathcal{T}_h$ of M, $T_h = \bigcup \mathcal{T}_h$, a tubular neighborhood U containing both M and T_h, we have the normal projection $a : U \to M$, and mappings $a_s : T_h \to M_h$ obtained by degree-s Lagrange interpolations of the component functions of a over each simplex. Let $\hat{\mathcal{T}}_h$ be the set of curved simplices that are the images under a_s of each simplex in $\mathcal{T}_h$. We define*

$$\mathcal{P}_r\Lambda^k(\hat{\mathcal{T}}_h) := (a_s^{-1})^*\mathcal{P}_r\Lambda^k(\mathcal{T}_h).$$

*We say that if $s = r$, the finite element space is **isoparametric**; if $s < r$, **subparametric**, and if $s > r$, **superparametric**. Similar definitions hold for the $\mathcal{P}^-$ spaces.*

Note that we do not enforce any more than C^0 continuity at the boundaries of the curved simplices (i.e. the surfaces M_h have no breaks, but may not have well-defined tangent hyperplanes on boundary faces of simplices). As noted before, however, continuity conditions enforced by the finite element spaces are compatible with this, just as they were for the piecewise linear case. Now, from $\mathcal{P}_r\Lambda^k(\hat{\mathcal{T}}_h)$, we can further pull these forms back to the surface M via the normal projections $(a|_{M_h})^{-1}$; this gives the injective morphisms $i_h^k : \Lambda_h^k \to H\Omega^k(M)$; it commutes with the differentials, since the pullbacks do.

The final piece in the theory we need to make this work is the bounded cochain operators. Motivated by the constructions in [2, 3], we shall assume the following:

Conjecture 5.5. *For $M_h = \bigcup \hat{\mathcal{T}}_h$, there exist bounded cochain projections*

$$\pi_h^{k\prime} : L^2\Omega^k(M_h) \to \Lambda_h^k$$

where (Λ_h, d) *is any of the* 2^{n-1} *discrete cochain complexes we have considered. Moreover, using the normal projection* $a|_{M_h}$*, we can pull forms back from* M_h *to* M*. Then* $\pi_h^k = \pi_h^{k\prime} \circ (a|_{M_h}^*) : L^2\Omega^k(M) \to \Lambda_h^k$ *are also bounded cochain operators* [24*, Theorem* 3.7]*, as required.*

This result does not yet appear in the literature, although it seems likely that a construction similar to that of Christiansen and Winther [11] is possible. Together with Conjecture 5.5, we shall assume that the analogue of Theorem 5.4 above holds for $\mathcal{T}$ a curved triangulation of a hypersurface. We note that these assumptions were also needed in [24, 25] for application to the surface finite element method.

5.4. Estimates for the Mixed Hodge Laplacian problem on manifolds

With this, we can then integrate the terms from [24, Example 4.6] to get the results for the parabolic equations (or, equivalently, add the variational crimes to [1, 22]). Let us consider now the mixed Hodge Laplacian problem on Riemannian hypersurfaces, considering the setup in the previous example. Namely, we consider $W^k = L^2\Omega^k(M)$, $V^k = H\Omega^k(M)$ as above, the approximating spaces $V_h^{k-1} = \mathcal{P}_{r+1}\Lambda^{k-1}(\hat{\mathcal{T}}_h)$ and $V_h^k = \mathcal{P}_r\Lambda^k(\hat{\mathcal{T}}_h)$, and finally the inclusion and projection morphisms as above (possibly with additional pullbacks for interpolation degree $s > 1$). Of course, as mentioned before, these are not the only ways of choosing the spaces, but we stay with, and make estimates based on, this choice for the remainder of this example (the same choice made in [24, Example 4.6]). For a function $\tilde{f} \in L^2\Omega^k(M)$, we have an approximate solution $(\sigma_h', u_h', p_h') \in i_h\mathfrak{X}_h'$ to the elliptic problem, on the true subcomplex i_hW_h (with modified inner product, as in the theory of §2.4). For $\tilde{f}$ sufficiently regular, and (σ, u, p) satisfying the regularity estimate [3, 22]

$$\|u\|_{H^{s+2}} + \|p\|_{H^{s+2}} + \|\sigma\|_{H^{s+1}} \le C\|\tilde{f}\|_{H^s}, \tag{5.3}$$

for $0 \le s \le s_{\max}$, then, since we are in the de Rham complex, where the cochain projections are W-bounded, we have the improved error estimates of Arnold, Falk, and Winther [3, §3.5 and p. 342] for the elliptic problem:

$$\|u - i_hu_h'\| + \|p - i_hp_h'\| \le Ch^{r+1}\|\tilde{f}\|_{H^{r-1}}, \tag{5.4}$$

$$\|d(u - i_hu_h')\| + \|\sigma - i_h\sigma_h'\| \le Ch^r\|\tilde{f}\|_{H^{r-1}}, \tag{5.5}$$

$$\|d(\sigma - i_h\sigma_h')\| \le Ch^{r-1}\|\tilde{f}\|_{H^{r-1}}. \tag{5.6}$$

We should also note that Arnold and Chen [1] prove that this also works for a nonzero harmonic part [1, Theorem 3.1]. Holst and Stern [24] augment these estimates to include the variational crimes; first they show, for hypersurfaces, the following result for the variational crime $\|I - J_h\|$:

Theorem 5.6 ([24], Theorem 4.4). *Let* M *be an oriented, compact* m*-dimensional hypersurface in* $\mathbb{R}^{m+1}$*, and* M_h *be a family of hypersurfaces lying in a tubular neighborhood* U *of* M *transverse to its fibers, such that* $\|\delta\|_{L^\infty(M_h)} \to 0$ *and* $\|\nu - \nu_h\|_{L^\infty(M_h)} \to 0$ *as* $h \to 0$*. Then for sufficiently small* h*,*

$$\|I - J_h\| \le C\Big(\|\delta\|_{L^\infty(M_h)} + \|\nu - \nu_h\|^2_{L^\infty(M_h)}\Big) \tag{5.7}$$

(*recall that* ν*, the normal to* M*, is extended to the whole tubular neighborhood, including* M_h*, by* $\nabla\delta$*).*

A result of Demlow [15, Proposition 2.3] states that, in the case that M_h is obtained by degree-s Lagrange interpolation, that $\|\delta\|_{L^\infty(M_h)} < Ch^{s+1}$ and $\|\nu - \nu_h\|_{L^\infty(M_h)} < Ch^s$. Thus, putting these results together, we have that

$$\|I - J_h\| \leq Ch^{s+1}. \tag{5.8}$$

For $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h) \in \mathfrak{X}_h$, the discrete solution to the elliptic problem now on the approximating complexes we have chosen, we have the estimates

$$\begin{aligned}&\|u - i_h\tilde{u}_h\| + \|p - i_h\tilde{p}_h\| + h\left(\|d(u - i_h\tilde{u}_h)\| + \|\sigma - i_h\tilde{\sigma}_h\|\right) + h^2\|d(\sigma - i_h\tilde{\sigma}_h)\| \\ \leq& C(h^{r+1}\|\tilde{f}\|_{H^{r-1}} + h^{s+1}\|\tilde{f}\|).\end{aligned} \tag{5.9}$$

We note the terms associated to the different powers of h above correspond exactly to the breakdown (4.14)-(4.16) above. For the elliptic projection in our problem, we also need to account for the nonzero harmonic part of the solution. Setting $\tilde{w} = P_{\mathfrak{H}}\tilde{u}$ and $\tilde{w}_h = \Pi_h\tilde{w}$, we have that our three additional terms (given by Theorem 2.8 above) are the corresponding best approximation error $\inf_{v\in V_h^k}\|\tilde{w} - v\|_V$, the $\|I - J_h\|$ term, and the data approximation $\|\tilde{w}_h - i_h^*\tilde{w}\|_h$. For the best approximation, we make use of our observation about the inequality (2.24), in which we may instead use the W-norm instead of the V-norm in the case that the projections are W-bounded, as they are here in the de Rham complex. Because $\tilde{w}$ is harmonic, it is smooth (and in particular, in H^{r+1}), so we may apply Theorem 5.4 to find that it is of order $Ch^{r+1}\|\tilde{w}\|_{H^{r+1}}$. The $\|I - J_h\|$ term is of order Ch^{s+1} by Theorem 5.6. Finally, by Theorem 2.10 above, we have that data approximation splits into the other two terms. Therefore, to summarize, we have

Theorem 5.7 (Estimates for the elliptic projection). *Consider $(\sigma(t), u(t))$, the solution to the parabolic problem* (3.6) *and $(\sigma_h(t), u_h(t))$ the semidiscrete solution in* (4.1) *above. Then we have the following estimates for the elliptic projection $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h)$:*

$$\begin{aligned}&\|u - i_h\tilde{u}_h\| + \|i_h\tilde{p}_h\| + h\left(\|d(u - i_h\tilde{u}_h)\| + \|\sigma - i_h\tilde{\sigma}_h\|\right) + h^2\|d(\sigma - i_h\tilde{\sigma}_h)\| \\ \leq& C\Big(h^{r+1}\left(\|\Delta u\|_{H^{r-1}} + \|\tilde{w}\|_{H^{r+1}}\right) + h^{s+1}\left(\|\Delta u\| + \|\tilde{w}\|\right)\Big).\end{aligned} \tag{5.10}$$

(We note $p = P_{\mathfrak{H}}(-\Delta u) = 0$.) We now would like use the our main parabolic estimates to analyze the analogous quantity

$$\|u(t) - i_h u_h(t)\| + h\left(\|d(u(t) - i_h u_h(t))\| + \|\sigma(t) - i_h\sigma_h(t)\|\right) + h^2\|d(\sigma(t) - i_h\sigma_h(t))\|, \tag{5.11}$$

and its integral, i.e. Bochner L^1 norm.

Theorem 5.8 (Main combined estimates for hypersurfaces). *Consider all terms involving the elliptic projection, $(\sigma(t), u(t))$, $(\sigma_h(t), u_h(t))$ defined as above, and suppose the regularity estimate* (5.3) *is satisfied. Then*

$$\begin{aligned}&\|u - i_h u_h\|_{L^1(W)} + h\left(\|d(u - i_h u_h)\|_{L^1(W)} + \|\sigma - i_h\sigma_h\|_{L^1(W)}\right) + h^2\|d(\sigma - i_h\sigma_h)\|_{L^1(W)} \\ \leq& C\Big[h^{r+1}\left((T+1)\left(\|\Delta u\|_{L^1(H^{r-1})} + \|\tilde{w}\|_{L^1(H^{r+1})}\right) + T\left(\|\Delta u_t\|_{L^1(H^{r-1})} + \|\tilde{w}_t\|_{L^1(H^{r+1})}\right)\right) \\ &+ h^{s+1}\left((T+1)\left(\|\Delta u\|_{L^1(W)} + \|\tilde{w}\|_{L^1(W)}\right) + T\left(\|\Delta u_t\|_{L^1(W)} + \|\tilde{w}_t\|_{L^1(W)}\right)\right)\Big].\end{aligned}$$

(We abbreviate $L^p(I,X)$ as $L^p(X)$.) The constants T, of course, can be further rolled into the constant C. We remark that in previous results, factors of T show up on the $\|\Delta u_t\|$ terms, and, heuristically speaking, this is due to the u_t being a physically different quantity, namely, a rate of change. However, the appearance of the factor of T on the $\|\Delta u\|$ comes from the harmonic approximation error $\tilde{p}_h$, which is, physically speaking, a harmonic source term. The details depend on the nature of the approximation operators Π_h.

Proof. By the triangle inequality, we have that (5.11) breaks up into something of the form (5.9) (taking $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h)$ to be elliptic projection with $\tilde{f} = -\Delta u(t)$ and $\tilde{p} = 0$; here $\tilde{f}$ is not to be confused with the *parabolic* source term $f(t)$) and

$$\|i_h\| \left(\|\theta(t)\|_h + h(\|\varepsilon(t)\|_h + \|du(t)\|_h) + h^2\|d\varepsilon(t)\|_h\right), \tag{5.12}$$

recalling the error quantities defined in (4.8)-(4.11). Now, substituting our estimates (4.14)-(4.16), we then have

$$\begin{aligned}
&\|\theta(t)\|_h \leq \|\rho_t\|_{L^1(W_h)} + \|\tilde{p}_h\|_{L^1(W_h)} + \|(\Pi_h - i_h^*)u_t\|_{L^1(W_h)} \\
\leq& C\Big(\|i_h\tilde{u}_{h,t} - u_t\|_{L^1(W)} + \|\tilde{p}_h\|_{L^1(W_h)} + \|I - J_h\|\,\|u_t\|_{L^1(W)} + \|(\Pi_h - i_h^*)u_t\|_{L^1(W_h)}\Big) \\
\leq& C_1 h^{r+1}\Big(\|\Delta u\|_{L^1(H^{r-1})} + \|\Delta u_t\|_{L^1(H^{r-1})} + \|\tilde{w}\|_{L^1(H^{r+1})} + \|\tilde{w}_t\|_{L^1(H^{r+1})}\Big) \\
&\quad + C_2 h^{s+1}\Big(\|\Delta u\|_{L^1(W)} + \|\Delta u_t\|_{L^1(W)} + \|\tilde{w}\|_{L^1(W)} + \|\tilde{w}_t\|_{L^1(W)}\Big).
\end{aligned} \tag{5.13}$$

For $\|d\theta\|_h + \|\varepsilon\|_h$, the computation is almost exactly the same, except with possibly different constants, to account for using L^2 Bochner norms, and that :

$$\begin{aligned}
&\|d\theta(t)\|_h + \|\varepsilon(t)\|_h \\
\leq& C\Big(\|i_h\tilde{u}_{h,t} - u_t\|_{L^2(W)} + \|\tilde{p}_h\|_{L^2(W_h)} + \|I - J_h\|\,\|u_t\|_{L^2(W)} + \|(\Pi_h - i_h^*)u_t\|_{L^2(W_h)}\Big) \\
\leq& C_3 h^{r+1}\Big(\|\Delta u\|_{L^2(H^{r-1})} + \|\Delta u_t\|_{L^2(H^{r-1})} + \|\tilde{w}\|_{L^2(H^{r+1})} + \|\tilde{w}_t\|_{L^2(H^{r+1})}\Big) \\
&\quad + C_4 h^{s+1}\Big(\|\Delta u\|_{L^2(W)} + \|\Delta u_t\|_{L^2(W)} + \|\tilde{w}\|_{L^2(W)} + \|\tilde{w}_t\|_{L^2(W)}\Big).
\end{aligned}$$

These terms are actually absorbed into the lower order terms by the extra factor of h, due to consisting entirely of the same order terms except using a different norm. However, the situation is slightly different for $\|d\varepsilon\|_h$; namely we use (5.5) to get a term of order h^r, and the d_h^* on the variational crime part also removing a factor of h:

$$\begin{aligned}
\|d\varepsilon(t)\|_h \leq& C\left(\|\psi_t\|_{L^2(W_h)} + \|d_h^*(\Pi_h - i_h^*)u_t\|_{L^2(W_h)}\right) \\
\leq& C\left(\|i_h\tilde{\sigma}_{h,t} - \sigma_t\|_{L^2(W)} + \|I - J_h\|\,\|\sigma_t\|_{L^2(W)} + \|d_h^*(\Pi_h - i_h^*)u_t\|_{L^2(W_h)}\right) \\
\leq& C_5 h^r\Big(\|\Delta u_t\|_{L^2(H^{r-1})} + \|\tilde{w}\|_{L^2(H^{r+1})}\Big) + C_6 h^s\Big(\|\Delta u\|_{L^2(W)} \\
&\quad + \|\Delta u_t\|_{L^2(W)} + \|\tilde{w}\|_{L^2(W)} + \|\tilde{w}_t\|_{L^2(W)}\Big).
\end{aligned}$$

However, we see that multiplying by h^2, this term also gets absorbed; thus we only need to consider the error from $\|d\theta\|_h$ in further calculation of the combined estimate. We have, thus

far:

$$\begin{aligned}
&\|u(t) - i_h u_h(t)\| + h\left(\|d(u(t) - i_h u_h(t))\| + \|\sigma(t) - i_h \sigma_h(t)\|\right) + h^2 \|d(\sigma(t) - i_h \sigma_h(t))\| \\
\leq & C_1 h^{r+1}\Big(\|\Delta u\|_{L^1(H^{r-1})} + \|\Delta u_t\|_{L^1(H^{r-1})} + \|\tilde{w}\|_{L^1(H^{r+1})} + \|\tilde{w}_t\|_{L^1(H^{r+1})}\Big) \\
&+ C_2 h^{s+1}\Big(\|\Delta u\|_{L^1(W)} + \|\Delta u_t\|_{L^1(W)} + \|\tilde{w}\|_{L^1(W)} + \|\tilde{w}_t\|_{L^1(W)}\Big) \\
&+ C\Big(h^{r+1}(\|\Delta u(t)\|_{H^{r-1}} + \|\tilde{w}(t)\|_{H^{r+1}}) + h^{s+1}(\|\Delta u(t)\| + \|\tilde{w}(t)\|)\Big). \qquad (5.14)
\end{aligned}$$

Integrating with respect to t from 0 to T, we find that the already-present Bochner norms are constant and thus introduce an extra factor of T. Absorbing the constants except T gives the result. □

With the above, we also have an interesting result regarding the optimal rate of convergence, which warrants stating as a separate theorem.

Theorem 5.9. *The optimal rate of convergence occurs when the polynomial degree of the finite element functions matches the degree of polynomials used to approximate the hypersurface (i.e., when $r = s$, above).*

This tells us, for example, that it is not beneficial to use finite elements of higher than linear order on, say, a (piecewise linear) triangulation. Similar observations have been made in [15] for the surface finite element method (SFEM). This is also consistent with observations in isogeometric analysis (IGA), which is a theoretical and computational framework for integrating finite element methods into CAD modeling software [12, 26] by using NURBS (Non-Uniform Rational B-Splines) as bases for finite element-type methods. SFEM and IGA can be viewed as distinct alternatives to the methods described in the present article. SFEM has been primarily used for two-dimensional problems, whereas IGA is particularly well-suited to structural mechanics problems. However, both are more broadly applicable, and can potentially be applied to problems on domains such as those considered in this article.

Finally, to put these estimates into some perspective and help develop some intuition for their meaning, we present the generalization of the estimates of Thomée.

Corollary 5.2 (Generalization of [1, 22, 35]). *Focusing on just the components u and σ separately, we have the following estimates (assuming the regularity estimates* (5.3) *are satisfied), and supposing $r = s$, i.e., the finite element spaces considered consist of polynomials of the same degree as the interpolation on the surface:*

$$\begin{aligned}
\|u(t) - i_h u_h(t)\| &\leq C h^{r+1}\left(\|u(t)\|_{H^{r+1}} + \int_0^t \left(\|u(s)\|_{H^{r+1}} + \|u_t(s)\|_{H^{r+1}}\right)\, ds\right), \\
\|\sigma(t) - i_h \sigma_h(t)\| &\leq C h^{r+1}\left(\|u(t)\|_{H^{r+2}} + \left(\int_0^t \left(\|u(s)\|^2_{H^{r+1}} + \|u_t(s)\|^2_{H^{r+1}}\right)\, ds\right)^{1/2}\right).
\end{aligned}$$

This easily leads to an estimate in a Bochner L^∞ norm (simply take the sup in the non-Bochner norm terms and $t = T$ in the integrals); this shows that the error in time is small at every $t \in I$. Similar estimates hold for $L^2(I, W)$ norms.

Proof. We consider the improved error estimate and variational crimes in u and σ separately. We first have, by expanding the terms in (4.20) as in the derivation of (5.13),

$$\|u(t) - i_h u_h(t)\| \le C\Big(\|u(t) - i_h \tilde{u}_h(t)\| + \|\theta(t)\|\Big) \le Ch^{r+1}\bigg(\|\Delta u(t)\|_{H^{r-1}} + \|\tilde{w}(t)\|_{H^{r+1}} + \int_0^t \left(\|\Delta u(s)\|_{H^{r-1}} + \|\Delta u_t(s)\|_{H^{r-1}} + \|\tilde{w}(s)\|_{H^{r-1}} + \|\tilde{w}_t(s)\|_{H^{r-1}}\right)\, ds\bigg).$$

The result follows by noting that $\|u\|_{H^{r+1}}$ includes estimates on all the second derivative terms in u, and $\tilde{w} = P_{\mathfrak{H}} u$, so those two norms can all be combined (with possibly different constants). Next, we consider σ. The improved error estimates [3, p. 342] imply that if we do not combine estimates involving du with those of σ for the modified solution, and $\tilde{f}$ is regular enough to use the H^r- rather than H^{r-1}-norm, then we can gain back one factor of h, so that it is of order h^{r+1} (rather than h^r as in (5.5)). On the other hand, the elliptic projection error $\|\varepsilon(t)\|$ still can be taken along with $\|d\sigma(t)\|$ and was of order h^{r+1} to begin with. Thus, applying (4.23), we have

$$\begin{aligned}
\|\sigma(t) - i_h \sigma_h(t)\| &\le C\Big(\|\sigma(t) - i_h \tilde{\sigma}_h(t)\| + \|\varepsilon(t)\| + \|du(t)\|\Big) \\
&\le Ch^{r+1}\bigg(\|\Delta u(t)\|_{H^r} + \|\tilde{w}(t)\|_{H^{r+1}} + \bigg[\int_0^t \big(\|\Delta u(s)\|^2_{H^{r-1}} \\
&\qquad + \|\Delta u_t(s)\|^2_{H^{r-1}} + \|\tilde{w}(s)\|^2_{H^{r-1}} + \|\tilde{w}_t(s)\|^2_{H^{r-1}}\big)\, ds\bigg]^{1/2}\bigg) \\
&\le Ch^{r+1}\bigg(\|u(t)\|_{H^{r+2}} + \bigg(\int_0^t \big(\|u(s)\|^2_{H^{r+1}} + \|u_t(s)\|^2_{H^{r+1}}\big)\, ds\bigg)^{1/2}\bigg),
\end{aligned}$$

where we have used the same consolidation techniques for the norms on Δu and $\tilde{w}$ into norms on u as before. □

We see the variational crimes (arising from the extra $\tilde{p}_h$) account for the sole additional term in the integrals. This cannot be improved without further information on the projections Π_h. Otherwise, for $r = 1$, which correspond to piecewise linear discontinuous elements for 2-forms (u), and piecewise quadratic elements for 1-forms (σ) with normal continuity (Raviart-Thomas elements), as studied by Thomée, we obtain the estimates he derived (and since the $\tilde{p}_h$ is not there in his case, we have that the extra terms with u do not appear under the integral sign).

6. Notes on Discretizing the Equation

We end with some remarks on how to actually prepare the equations for actual numerical computation. We do not give an error analysis for a fully discretized problem, saving that for future work. We return to the evolution equation for both σ and u, (3.6) above, which we recall here:

$$\begin{aligned}
&\langle \sigma_t, \omega\rangle + \langle d\sigma, d\omega\rangle = \langle f, d\omega\rangle, && \forall\, \omega \in V^{k-1}, \quad t \in I, \\
&\langle u_t, \varphi\rangle + \langle d\sigma, \varphi\rangle + \langle du, d\varphi\rangle = \langle f, \varphi\rangle, && \forall\, \varphi \in V^{k}, \quad t \in I, \\
&u(0) = g.
\end{aligned} \tag{6.1}$$

Given $S_h \subseteq V^k = H\Omega^2(M)$ and $H_h \subseteq V^{k-1} = H\Omega^1(M)$, we choose bases, and use the semidiscrete equations (1.4), which we recall here (setting U to be the coefficients of u_h in the

basis for S_h, and Σ to be the coefficients of σ_h in the basis for H_h)

$$\frac{d}{dt}\begin{pmatrix} D & -B^T \\ 0 & A \end{pmatrix}\begin{pmatrix} \Sigma \\ U \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -B & -K \end{pmatrix}\begin{pmatrix} \Sigma \\ U \end{pmatrix} + \begin{pmatrix} 0 \\ F \end{pmatrix}. \tag{6.2}$$

This may be discretized via standard methods for ODEs. For example, if we use the backward Euler method, we consider sequences (Σ^n, U^n) in time, and then rewrite the derivative instead as a finite difference, evaluating the vector field portion on the right side at timestep $n+1$, taking $M = \begin{pmatrix} D & -B^T \\ 0 & A \end{pmatrix}$:

$$\frac{1}{\Delta t} M \left(\begin{pmatrix} \Sigma^{n+1} \\ U^{n+1} \end{pmatrix} - \begin{pmatrix} \Sigma^{n} \\ U^{n} \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ -B & -K \end{pmatrix}\begin{pmatrix} \Sigma^{n+1} \\ U^{n+1} \end{pmatrix} + \begin{pmatrix} 0 \\ F^{n+1} \end{pmatrix},$$

or

$$\left(M + \Delta t \begin{pmatrix} 0 & 0 \\ B & K \end{pmatrix} \right) \begin{pmatrix} \Sigma^{n+1} \\ U^{n+1} \end{pmatrix} = M \begin{pmatrix} \Sigma^{n} \\ U^{n} \end{pmatrix} + \Delta t \begin{pmatrix} 0 \\ F^{n+1} \end{pmatrix}.$$

We now have written the system as a sparse matrix times the unknown (Σ^{n+1}, U^{n+1}). This allows us to solve the system directly using sparse matrix algorithms without explicitly inverting any matrices, making the iterations efficient.

7. Conclusion and Future Directions

We have seen that the abstract theory of Hilbert complexes, as detailed by Arnold, Falk, and Winther [3], and Bochner spaces, as detailed in Gillette, Holst, and Zhu [22] and Arnold and Chen [1], has been very useful in clarifying the important aspects of elliptic and parabolic equations. The mixed formulation gives great insight into questions of existence, uniqueness, and stability of the numerical methods (linked by the cochain projections π_h). The method of Thomée [35] allows us to leverage the existing theory for elliptic problems to apply to parabolic problems, taking care of the remaining error terms by the use of differential inequalities and Grönwall estimates (in the important error evolution equations (4.12) above). Incorporating the analysis of variational crimes allow us to carry this theory over to the case of surfaces and their approximations.

We remark on some possible future directions for this work. Some existing surface finite elements for parabolic equations have been studied by Dziuk and Elliott [18], and much other work by Dziuk, Elliott, Deckelnick [13, 14], which actually treat the case of an evolving surface, and treat a nonlinear equation, the mean curvature flow. Generally speaking, this translates to an additional time dependence for evolving metric coefficients, and a logical place to start is in the Thomée error evolution equations (4.12). Nonlinear evolution equations for evolving metrics also suggests the Ricci flow [9, 10, 30], instrumental in showing the Poincaré conjecture. The challenge there, besides nonlinearity, is that tensor equations do not necessarily fit in the framework for FEEC. On the other hand, the Yamabe flow [34], which solves for a conformal factor for the metric (and is equivalent to the Ricci flow in dimension 2) suggests an interesting nonlinear scalar evolution equation for which this analysis may be useful.

Gillette, Holst, and Zhu [22] also analyzed hyperbolic equations in this framework, and it would be interesting and useful to analyze methods on surfaces (including the evolving case), as well as taking a more integrated approach in spacetime. This is usually taken care of

using the discrete exterior calculus (DEC), the finite-difference counterpart to FEEC to analyze hyperbolic equations [29].

Acknowledgments. We wish to thank both referees for their careful review of the manuscript, and for the guidance and suggestions they gave which substantially improved the paper. MH was supported in part by NSF Awards 1217175, 1262982, and 1318480. CT was supported in part by NSF Award 1217175.

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