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FINITE ELEMENT EXTERIOR CALCULUS FOR PARABOLIC EVOLUTION PROBLEMS ON RIEMANNIAN HYPERSURFACES*

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Abstract

Over the last ten years, Finite Element Exterior Calculus (FEEC) has been developed as a general framework for linear mixed variational problems, their numerical approximation by mixed methods, and their error analysis. The basic approach in FEEC, pioneered by Arnold, Falk, and Winther in two seminal articles in 2006 and 2010, interprets these problems in the setting of *Hilbert complexes*, leading to a more general and complete understanding. Over the last five years, the FEEC framework has been extended to a broader set of problems. One such extension, due to Holst and Stern in 2012, was to problems with *variational crimes*, allowing for the analysis and numerical approximation of linear and geometric elliptic partial differential equations on Riemannian manifolds of arbitrary spatial dimension. Their results substantially generalize the existing surface finite element approximation theory in several respects. In 2014, Gillette, Holst, and Zhu extended FEEC in another direction, namely to parabolic and hyperbolic evolution systems by combining the FEEC framework for elliptic operators with classical approaches for parabolic and hyperbolic operators, by viewing solutions to the evolution problem as lying in Bochner spaces (spaces of Banach-space valued parametrized curves). Related work on developing an FEEC theory for parabolic evolution problems has also been done independently by Arnold and Chen. In this article, we extend the work of Gillette-Holst-Zhu and Arnold-Chen to evolution problems on Riemannian manifolds, through the use of framework developed by Holst and Stern for analyzing variational crimes. We establish *a priori* error estimates that reduce to the results from earlier work in the flat (non-criminal) setting. Some numerical examples are also presented.

Mathematics subject classification: 65M15, 65M60, 53C44
Key words: FEEC, Elliptic equations, Evolution equations, Approximation theory, Inf-sup conditions, *A priori* estimates, Variational crimes, Equations on manifolds

1. Introduction

Arnold, Falk, and Winther [2, 3] introduced Finite Element Exterior Calculus (FEEC) as a general framework for linear mixed variational problems, their numerical approximation by mixed methods, and their error analysis. They recast these problems using the ideas and tools of *Hilbert complexes*, leading to a more complete understanding. Subsequently, Holst and Stern [24] extended the Arnold–Falk–Winther framework to include *variational crimes*, allowing for the analysis and numerical approximation of linear and geometric elliptic partial differential equations on Riemannian manifolds of arbitrary spatial dimension, generalizing the

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existing surface finite element approximation theory in several directions. Gillette, Holst, and Zhu [22] extended FEEC in another direction, namely to parabolic and hyperbolic evolution systems by combining recent work on FEEC for elliptic problems with a classical approach of Thomée [35] to solving evolution problems using semi-discrete finite element methods, by viewing solutions to the evolution problem as lying in Bochner spaces (spaces of Banach-space valued parametrized curves). Arnold and Chen [1] independently developed related work, for generalized Hodge Laplacian parabolic problems for differential forms of arbitrary degree, and Holst, Mihalik, and Szypowski [23] consider similar work in adaptive finite element methods.

In this article, we aim to combine the approaches of the above articles, extending the work of Gillette, Holst, and Zhu [22] and Arnold and Chen [1] to evolution problems in abstract Hilbert complexes by using the framework of Holst and Stern [24]. We then apply the results to parabolic problems on Riemannian hypersurfaces approximated by piecewise polynomial curved triangulations in a tubular neighborhood, using piecewise polynomial finite element spaces. As in earlier literature on finite elements on approximating surfaces by Dziuk [17], Dziuk and Demlow [16], and Demlow [15], the error splits into a PDE approximation term and a surface approximation term. An interesting result that follows is that the optimal rate of convergence occurs when the polynomial degree of both the approximating surfaces and the finite element spaces are the same (the isoparametric case). Similar observations have been made for the surface finite element method [15] and in the isogeometric analysis literature [12, 26].

1.1. The Hodge heat equation and its mixed form

We now motivate our problem with a concrete example. We consider an evolution equation for differential forms on a manifold, and then we rephrase it as a mixed problem as an intermediate step toward semidiscretization using mixed finite element methods. We then see how this allows us to leverage existing *a priori* error estimates for parabolic problems, and see how it fits in the framework of Hilbert complexes.

Let M be a smooth compact oriented Riemannian n-manifold without boundary embedded in $\mathbb{R}^{n+1}$. The **Hodge heat equation** is to find time-dependent k-form

$$u : M \times [0,T] \to \Lambda^k(M)$$

(where $\Lambda^k(M)$ denotes the bundle of alternating k-tensors on M) such that

$$\begin{aligned} &u_t - \Delta u = u_t + (\delta d + d\delta)u = f \quad && \text{in } M, \quad \text{for } t > 0 \\ &u(\cdot, 0) = g && \text{in } M. \end{aligned} \tag{1.1}$$

where g is an initial k-form, and f, a possibly time-dependent k-form, is a source term. Note that no boundary conditions are needed since $\partial M = \emptyset$. This is the problem studied by Arnold and Chen [1], and in the case $k = n$, one of the problems studied by Gillette, Holst, and Zhu [22], building upon work in special cases for domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ by Johnson and Thomée [27, 35]

For the stability of the numerical approximations with the methods of [25] and [3], we recast the problem in mixed form, converting the problem into a system of differential equations. Motivating the problem by setting $\sigma = \delta u$ (recall that for the Dirichlet problem and $k = n$, δ here corresponds to the *negative* of the gradient in Euclidean space, and is the adjoint of d, corresponding to the divergence), and taking the adjoint, we have

$$\begin{aligned} &\langle \sigma, \omega \rangle - \langle u, d\omega \rangle = 0, && \forall\, \omega \in H\Omega^{k-1}(M), \quad t > 0, \\ &\langle u_t, \varphi \rangle + \langle d\sigma, \varphi \rangle + \langle du, d\varphi \rangle = \langle f, \varphi \rangle, && \forall\, \varphi \in H\Omega^k(M) \qquad t > 0, \\ &u(0) = g \end{aligned} \tag{1.2}$$