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UNCONDITIONALLY SUPERCLOSE ANALYSIS OF A NEW MIXED FINITE ELEMENT METHOD FOR NONLINEAR PARABOLIC EQUATIONS*

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Abstract

This paper develops a framework to deal with the unconditional superclose analysis of nonlinear parabolic equation. Taking the finite element pair $Q_{11}/Q_{01}\times Q_{10}$ as an example, a new mixed finite element method (FEM) is established and the τ-independent superclose results of the original variable u in H^1-norm and the flux variable $\vec{q}=-a(u)\nabla u$ in L^2-norm are deduced (τ is the temporal partition parameter). A key to our analysis is an error splitting technique, with which the time-discrete and the spatial-discrete systems are constructed, respectively. For the first system, the boundedness of the temporal errors are obtained. For the second system, the spatial superclose results are presented unconditionally, while the previous literature always only obtain the convergent estimates or require certain time step conditions. Finally, some numerical results are provided to confirm the theoretical analysis, and show the efficiency of the proposed method.

Mathematics subject classification: 65N15, 65N30
Key words: Nonlinear parabolic equation, Mixed FEM; Time-discrete and spatial-discrete systems, τ-independent superclose results.

1. Introduction

Let $\Omega\subset\mathbb{R}^2$ be a rectangle with boundary $\partial\Omega$ and $0<T<\infty$. We develop and analyze a mixed FEM to the following time-dependent nonlinear parabolic equation:

$$\begin{cases} u_t-\nabla\cdot(a(u)\nabla u)=f(X,t), & (X,t)\in\Omega\times(0,T],\\ u=0, & (X,t)\in\partial\Omega\times(0,T],\\ u(X,0)=u_0(X), & X\in\Omega, \end{cases} \tag{1.1}$$

where $X=(x,y)$, $a(u)$ and $f(X,t)$ are smooth functions. Assume that there exist constants μ, M, B such that $0<\mu\le a(u)\le M$, $|a'(u)+a''(u)|\le B$. For the nonlinear problem of (1.1), [1] constructed the linearized Galerkin FEM and derived optimal error of order $O(h^2+\tau^2)$ in L^2-norm. With the linearized Galerkin FEMs, [2] and [3] discussed three-level Galerkin method and implicit-explicit multistep FEMs, and obtained optimal order error estimates, respectively. For other nonlinear problems, numerious efforts have been devoted to the development of efficient numerical schemes, such as the nonlinear parabolic integro-differential equations [4-6], nonlinear Schrödinger equations [7-10], Navier-Stokes equations [11-13] and others [14-18].

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It is known that the approximating spaces should satisfy the so-called Babuska-Brezzi condition in the usual mixed FEMs. In order to make the requirement to be satisfied easier, a mixed finite element form was established in [19] for second elliptic problems, in which the two spaces just need to fulfill a very simple inclusion relationship. Motivated by this work, the nonconforming pair $EQ_1^{rot}/Q_{10} \times Q_{01}$ was used to research a linear Sobolev equation and optimal error estimates and superclose results were received in [20]. For the linear parabolic problem, [21] deduced optimal error estimates based on the triangular nonconforming finite element pair $P_1/P_0 \times P_0$, and [22] showed the supercloseness as well as the extrapolation results with the nonconforming element pair $EQ_1^{rot}/Q_{10} \times Q_{01}$ of [20]. Note that [23,24] discussed the linear elasticity problem and the nonlinear Schrödinger equation with conforming finite element pairs, respectively.

Generally speaking, to deduce optimal error estimates of linearized Galerkin FEMs, one may use mathematical induction with an inverse inequality to bound the numerical solution in L^∞ norm, such as

$$\|U_h^n - R_h u^n\|_{L^\infty} \leq Ch^{-\frac{d}{2}} \|U_h^n - R_h u^n\|_0 \leq Ch^{-\frac{d}{2}}(h^{r+1} + \tau^m), \tag{1.2}$$

Here and later, U_h^n and u^n are the finite element approximation and the exact solution at time t^n, respectively, and R_h is a certain projection operator, C is a positive constant independent of τ and h. The inequality (1.2) results in the time-step restriction, and extremely time-consuming in practical computations see, e.g., [3-18,24,25]. However, it has been shown that the time restriction may not be necessary in many cases (see $[26-33]$). Not long ago, a new error analysis technique was proposed by [26] (also see [27]) for a Joule heating system with a standard Galerkin FEM, which splitted the numerical error into two parts, the spatial error and the temporal error. Then, the estiamte of (1.2) can be replaced by

$$\|U_h^n - R_h U^n\|_{L^\infty} \leq Ch^{-\frac{d}{2}} \|U_h^n - R_h U^n\|_0 \leq Ch^{-\frac{d}{2}} h^{r+1}, \tag{1.3}$$

where U^n is the time-discrete solution. Therefore, the bouneness of U_h^n can be deduced without any time-restriction. Consequently, [28-31] applied this idea to investigate various nonlinear problems and obtained the unconditional error estimates, respectively. But in the above studies, they only focused on the analysis of time-independent error estimates for the lineared Galerkin FEMs. Recently, [32] studied a mixed finite element scheme for the nonlinear Sobolev equation, and obtain the unconditionaliy superclose and superconvergent results by avoiding the estimate of the numerical solution in L^∞-norm. Of course, the method can't be used in this equation of (1.1). [33] derived the unconditionally superconvergent results for nonlinear parabolic equation with nonconforming EQ_1^{rot} elemrnt. In this paper, we study the linearized mixed finite element scheme for problem (1.1) with element pair $Q_{11}/Q_{01} \times Q_{10}$, and deduce the τ-independent superclose results through rigorous analysis.

The rest of the paper is organized as follows. In Section 2, the linearized time-discrete system is presented and the boundedness of the numerical solution in L^∞ norm for the original variable u and the flux variable $\vec{q} = -a(u)\nabla u$ are deduced, which will play an important role in the superclose analysis. In Section 3, we develop the new mixed finite element scheme and some notations. In Section 4, we give the linearized FEM for the spatial-discrete system and derive the corresponding superclose estimates of order $O(h^2 + \tau^2)$ unconditionally. In Section 5, some numerical results are provided to verify the theoretical analysis.

2. Error Analysis for the Time-Discrete System

Let $\vec{q} = -a(u)\nabla u$. We can rewrite problem (1.1) as

$$\begin{cases} u_t + \nabla\cdot\vec{q} = f, & (X,t)\in\Omega\times(0,T],\\ \vec{q} + a(u)\nabla u = 0, & (X,t)\in\Omega\times(0,T],\\ u(X,t) = 0, & (X,t)\in\partial\Omega\times(0,T],\\ u(X,0) = u_0(X), & X\in\Omega. \end{cases} \tag{2.1}$$

For positive integer N, let $0 = t_0 < t_1 < \ldots < t_N = T$ be a given partition of $[0,T]$ with step length $\tau = T/N$, $t_{n-\frac{1}{2}} = \frac{1}{2}(t_n + t_{n-1})$, and $t_n = n\tau, n = 0,1,\ldots,N$.

Let $u^n = u(X,t_n)$ $(n = 0,1,\ldots,N)$, we define

$$\overline{\partial}_t u^n = (u^n - u^{n-1})/\tau, \quad \widehat{u}^n = \frac{1}{2}(u^n + u^{n-1}), \quad n = 1,2,\cdots,N,$$
$$\overline{u}^n = \frac{1}{2}(3u^{n-1} - u^{n-2}), \quad n = 2,\cdots,N.$$

For $n \geq 2$, With above notations, Eq. (2.1) can be rewritten as

$$\begin{cases} \overline{\partial}_t u^n + \nabla\cdot\widehat{\vec{q}}^{\,n} = f^{n-\frac{1}{2}} + R_1^n + \nabla\cdot R_2^n, & (X,t)\in\Omega\times(0,T],\\ \widehat{\vec{q}}^{\,n} + a(\overline{u}^n)\nabla\widehat{u}^n = R_2^n + R_3^n, & (X,t)\in\Omega\times(0,T],\\ u^n = 0, & (X,t)\in\partial\Omega\times(0,T],\\ u^0 = u_0(X), & X\in\Omega, \end{cases} \tag{2.2}$$

where $R_1^n = \overline{\partial}_t u^n - u_t^{n-\frac{1}{2}}$, $R_2^n = \widehat{\vec{q}}^{\,n} - \vec{q}^{\,n-\frac{1}{2}}$, $R_3^n = a(\overline{u}^n)\nabla\widehat{u}^n - a(u^{n-\frac{1}{2}})\nabla u^{n-\frac{1}{2}}$.

Then we can formulate the following time-discrete system for (2.1): Find $\{U^n, \widehat{\vec{Q}}^{\,n}\}$ $(n \geq 2)$ such that

$$\begin{cases} \overline{\partial}_t U^n + \nabla\cdot\widehat{\vec{Q}}^{\,n} = f^{n-\frac{1}{2}}, & (X,t)\in\Omega\times(0,T],\\ \widehat{\vec{Q}}^{\,n} + a(\overline{U}^n)\nabla\widehat{U}^n = 0, & (X,t)\in\Omega\times(0,T],\\ U^n = 0, & (X,t)\in\partial\Omega\times(0,T],\\ U^0 = u_0(X), & X\in\Omega, \end{cases} \tag{2.3}$$

For $n = 1$, we calculate $\{U^1, \widehat{\vec{Q}}^{\,1}\}$ in the following two steps (see [1]).

Step 1:

$$\begin{cases} \dfrac{U^{1,0} - U^{1,0}}{\tau} + \dfrac{\nabla\cdot\vec{Q}^0 + \nabla\cdot\vec{Q}^{1,0}}{2} = f^{\frac{1}{2}}, & (X,t)\in\Omega\times(0,T],\\ \dfrac{\vec{Q}^0 + \vec{Q}^{1,0}}{2} + a(U^0)\dfrac{\nabla U^0 + \nabla U^{1,0}}{2} = 0, & (X,t)\in\Omega\times(0,T],\\ U^{1,0} = 0, & (X,t)\in\partial\Omega\times(0,T],\\ U^0 = u_0(X), & X\in\Omega, \end{cases} \tag{2.4}$$

together with

$$\begin{cases} \overline{\partial}_t u^1 + \nabla\cdot\widehat{\vec{q}}^{\,1} = f^{\frac{1}{2}} + R_1^0 + \nabla\cdot R_2^0, & (X,t)\in\Omega\times(0,T],\\ \widehat{\vec{q}}^{\,1} + a(u^0)\nabla\widehat{u}^1 = R_2^0 + R_3^0, & (X,t)\in\Omega\times(0,T],\\ u^1 = 0, & (X,t)\in\partial\Omega\times(0,T],\\ u^0 = u_0(X), & X\in\Omega, \end{cases} \tag{2.5}$$

where $R_1^0=\overline{\partial}_t u^1-u_t^{\frac{1}{2}}$, $R_2^0=\widehat{\vec{q}}^1-\vec{q}^{\frac{1}{2}}$, $R_3^0=a(u^0)\nabla\widehat{u}^1-a(u^{\frac{1}{2}})\nabla u^{\frac{1}{2}}$.

Step 2:

$$\begin{cases}\overline{\partial}_t U^1+\nabla\cdot\widehat{\vec{Q}}^1=f^{\frac{1}{2}}, & (X,t)\in\Omega\times(0,T],\\ \widehat{\vec{Q}}^1+a(\frac{U^0+U^{1,0}}{2})\nabla\widehat{U}^1=0, & (X,t)\in\Omega\times(0,T],\\ U^1=0, & (X,t)\in\partial\Omega\times(0,T],\\ U^0=u_0(X), & X\in\Omega,\end{cases}\tag{2.6}$$

together with

$$\begin{cases}\overline{\partial}_t u^1+\nabla\cdot\widehat{\vec{q}}^1=f^{\frac{1}{2}}+R_1^1+\nabla\cdot R_2^1, & (X,t)\in\Omega\times(0,T],\\ \widehat{\vec{q}}^1+a(\widehat{u}^1)\nabla\widehat{u}^1=R_2^1+R_3^1, & (X,t)\in\Omega\times(0,T],\\ u^1=0, & (X,t)\in\partial\Omega\times(0,T],\\ u^0=u_0(X), & X\in\Omega,\end{cases}\tag{2.7}$$

where $R_1^1=\overline{\partial}_t u^1-u_t^{\frac{1}{2}}$, $R_2^1=\widehat{\vec{q}}^1-\vec{q}^{\frac{1}{2}}$, $R_3^1=a(\widehat{u}^1)\nabla\widehat{u}^1-a(u^{\frac{1}{2}})\nabla u^{\frac{1}{2}}$.

Let $e^n=u^n-U^n$, $e^{1,0}=u^1-U^{1,0}$ and $\sigma^n=q^n-Q^n$, $\sigma^{1,0}=u^1-U^{1,0}$. Then we have the following important theorem.

Theorem 2.1. *Let that $\{u^n,\widehat{\vec{q}}^n\}$ and $\{U^n,\widehat{\vec{Q}}^n\}$ $(n\geq 2)$ be the solutions of* (2.2) *and* (2.3) *respectively. And the time-discrete system* (2.7) *and* (2.6) *have the solutions $\{u^1,\widehat{\vec{q}}^1\}$ and $\{U^1,\widehat{\vec{Q}}^1\}$ respectively. Assume that $u\in L^2(0,T;H^3(\Omega)\cap H_0^1(\Omega))$, $u_t\in L^\infty(0,T;H^2(\Omega))$, $u_{tt}\in L^2(0,T;H^2(\Omega))$, $u_{ttt}\in L^2(0,T;L^2(\Omega))$. Then for $n=1,\cdots,N$, there exists $\tau_0>0$, such that $\tau\leq\tau_0$,*

$$\|e^n\|_1+\tau|e^n|_2+\|e^{1,0}\|_0+\tau^{\frac{1}{2}}\|e^{1,0}\|_1+\tau|e^{1,0}|_2+\|\widehat{\vec{\sigma}}^n\|_0+\tau|\widehat{\vec{\sigma}}^n|_1+\tau\|\widehat{\vec{\sigma}}^{1,0}\|_1\leq C_0\tau^2,\tag{2.8}$$

$$\|\overline{\partial}_t U^n\|_2+\|U^n\|_2+\|\overline{\partial}_t U^{1,0}\|_2+\|U^{1,0}\|_2+\|\overline{\partial}_t\widehat{\vec{Q}}^n\|_1+\|\widehat{\vec{Q}}^n\|_1+\|\widehat{\vec{Q}}^{1,0}\|_1\leq C_0.\tag{2.9}$$

Proof. The results can be proved by using a similar technique given in [33]. □

3. Mixed Finite Element Spaces

Our domain Ω is a rectangle whose edges parallel to the x and y-axis. Moreover, T_h is a regular rectangular subdivision of Ω. The associated finite element spaces V_h and $\vec{W}_h$ are defined by

$$V_h=\Big\{v;v|_K\in Q_{11}(K),\forall K\in T_h\Big\},\quad V_0^h=\Big\{v;v\in V_h,v|_{\partial\Omega}=0\Big\},$$
$$\vec{W}_h=\Big\{\vec{w}=(w^1,w^2)\in(L^2(\Omega))^2;\vec{w}|_K\in Q_{01}(K)\times\in Q_{10}(K),\forall K\in T_h\Big\},$$

where $Q_{ij}=span\{x^ry^s,0\leq r\leq i,0\leq s\leq j\}$.

Then for all $v\in H^2(\Omega),\vec{w}_h=(w_1,w_2)\in(H^1(\Omega))^2$, we define the interpolation operators I_h,Π_h as

$$I_h:\ \ v\in H^2(\Omega)\to I_hv\in V_h,\quad I_h|_K=I_K,\quad I_Kv(a_i)=v(a_i),\quad i=1,2,3,4,$$
$$\Pi_h:\ \ \vec{q}\in(H^1(\Omega))^2\to\Pi_h\vec{q}\in\vec{W}_h,\quad \Pi_h|_K=\Pi_K,\quad \int_{l_i}(\vec{q}-\Pi_K\vec{q})\cdot\vec{\tau}_i ds=0,$$

respectively, where $\vec{\tau}_i$ is the unit tangent vector of l_i.

For $u \in H^3(\Omega)$, $\vec{q} \in (H^2(\Omega))^2$, there hold (see [35])

$$(\nabla(u - I_h u), \nabla v) \le Ch^2 \|u\|_3 \|v\|_1, \qquad \forall v_h \in V_h, \tag{3.1}$$

$$(\vec{q} - \Pi_h \vec{q}, \vec{w}_h) \le Ch^2 \|\vec{q}\|_2 \|\vec{w}_h\|_0, \qquad \forall \vec{w}_h \in \vec{W}_h. \tag{3.2}$$

4. Superclose Analysis for the Spatial-discrete System

In this section, we will establish a τ-independent estimate. The fully-discrete approximation to (2.1) reads as: for $n \ge 2$, find $\{U_h^n, \widehat{\vec{Q}}_h^n\} \in V_h \times \overrightarrow{W}_h$, such that

$$\begin{cases} (\overline{\partial}_t U_h^n, v_h) - (\widehat{\vec{Q}}_h^n, \nabla v_h) = (f^{n-\frac{1}{2}}, v_h), & \forall v_h \in V_h, \\ (\widehat{\vec{Q}}_h^n, \vec{w}_h) + (a(\overline{U}_h^n)\nabla \widehat{U}_h^n, \vec{w}_h) = 0, & \forall \overrightarrow{w}_h \in \vec{W}_h, \\ U_h^0 = I_h u_0(X), & X \in \Omega. \end{cases} \tag{4.1}$$

For $n = 1$, $\{U_h^1, \widehat{\vec{Q}}_h^1\}$ will be determined by

$$\begin{cases} (\overline{\partial}_t U_h^1, v_h) - (\widehat{\vec{Q}}_h^1, \nabla v_h)_h = (f^{\frac{1}{2}}, v_h), & \forall v_h \in V_h, \\ (\widehat{\vec{Q}}_h^1, \vec{w}_h) + \Big(a(\frac{U_h^{1,0}+U_h^0}{2})\nabla \widehat{U}_h^1, \vec{w}_h\Big) = 0, & \forall \overrightarrow{w}_h \in \vec{W}_h, \\ U_h^0 = I_h u_0(X), & X \in \Omega. \end{cases} \tag{4.2}$$

together with

$$\begin{cases} \Big(\frac{U_h^{1,0}-U_h^0}{\tau}, v_h\Big) - \Big(\frac{\vec{Q}_h^{1,0}+\vec{Q}_h^0}{2}, \nabla v_h\Big)_h = (f^{\frac{1}{2}}, v_h), & \forall v_h \in V_h, \\ \Big(\frac{\vec{Q}_h^{1,0}+\vec{Q}_h^0}{2}, \vec{w}_h\Big) + \Big(a(U_h^0)\frac{\nabla U_h^{1,0}+\nabla U_h^0}{2}, \vec{w}_h\Big) = 0, & \forall \overrightarrow{w}_h \in \vec{W}_h, \\ U_h^0 = I_h U^0(X), & X \in \Omega. \end{cases} \tag{4.3}$$

Theorem 4.1. *Assume that $f(X,t)$ and $u_0(X)$ are known smooth functions. Then the system* (4.1)-(4.3) *is uniquely solvable.*

Proof. In terms of the bases $\{\phi_i\}_{i=1}^{\sigma_1}$ for V_h and $\{\vec{\psi}_i\}_{i=1}^{\sigma_2}$ for $\vec{W}_h$, we can suppose that

$$U_h^n = \sum_{i=1}^{\sigma_1} h_i(t^n)\phi_i, \quad \vec{Q}_h^n = \sum_{i=1}^{\sigma_2} g_i(t^n)\vec{\psi}_i.$$

On one hand, choosing $v_h = \phi_j, \vec{w}_h = \nabla\phi_j$, summing the first equation and the second equation of (4.1), then we have

$$(2M + \tau A)H^n = 2MH^{n-1} - \tau A H^{n-1} + 2\tau F^{n-\frac{1}{2}} \tag{4.4}$$

where $H^n = [h_i(t^n)]_{\sigma_1\times 1}, M = [(\phi_i, \phi_j)]_{\sigma_1\times\sigma_1}$, $F^{n-\frac{1}{2}} = [(f(t^{n-\frac{1}{2}}), \nabla\cdot\vec{\psi}_j)]_{\sigma_1\times 1}$, and

$$A = \Bigg[\Bigg(a^{\frac{1}{2}}\Bigg(\frac{3}{2}\sum_{i=1}^{\sigma_1} h_i(t^{n-2})\phi_i - \frac{1}{2}\sum_{i=1}^{\sigma_1} h_i(t^{n-1})\phi_i\Bigg)\nabla\phi_i, \\ a^{\frac{1}{2}}\Bigg(\frac{3}{2}\sum_{i=1}^{\sigma_1} h_i(t^{n-2})\phi_i - \frac{1}{2}\sum_{i=1}^{\sigma_1} h_i(t^{n-1})\phi_i\Bigg)\nabla\phi_j\Bigg)\Bigg]_{\sigma_1\times\sigma_1}.$$

In view of the positive definite matrix A, M and the initial value H^1 determined by (4.2)-(4.3), we have that H^n is determined uniquely.

On the other hand, choosing $\vec{w}_h = \vec{\psi}_i$ in the second equation of (4.1), it follows that

$$NG^n = -NG^{n-1} - BH^{n-1} - BH^n \tag{4.5}$$

where $G^n = [g_i(t^n)]_{\sigma_2\times 1}, N = [(\vec{\psi}_i, \vec{\psi}_i)]_{\sigma_2\times\sigma_2}$, and

$$B = \Big[\Big(a^{\frac{1}{2}}\Big(\frac{3}{2}\sum_{i=1}^{\sigma_1} h_i(t^{n-2})\phi_i - \frac{1}{2}\sum_{i=1}^{\sigma_1} h_i(t^{n-1})\phi_i\Big)\nabla\phi_i,$$
$$a^{\frac{1}{2}}\Big(\frac{3}{2}\sum_{i=1}^{\sigma_1} h_i(t^{n-2})\phi_i - \frac{1}{2}\sum_{i=1}^{\sigma_1} h_i(t^{n-1})\phi_i\Big)\vec{\psi}_i\Big)\Big]_{\sigma_2\times\sigma_1}.$$

Because of the positive definite matrix N and the initial value G^1 determined by (4.2)-(4.3), we have that G^n is determined uniquely. It can be followed that the system (4.1)-(4.3) has unique solutions for $t \in (0, T]$. The proof is completed. □

To alleviate the notations, we write

$$\begin{aligned}
&U^n - U_h^n = U^n - I_hU^n + I_hU^n - U_h^n \triangleq \psi^n + \varphi^n,\\
&u^n - U_h^n = u^n - I_hu^n + I_hu^n - U_h^n \triangleq \eta^n + \xi^n,\\
&\vec{Q}^n - \vec{Q}_h^n = \vec{Q}^n - \Pi_h\vec{Q}^n + \Pi_h\vec{Q}^n - \vec{Q}_h^n \triangleq \vec{\rho}^n + \vec{\zeta}^n,\\
&\vec{q}^n - \vec{Q}_h^n = \vec{q}^n - \Pi_h\vec{q}^n + \Pi_h\vec{q}^n - \vec{Q}_h^n \triangleq \vec{r}^n + \vec{\theta}^n,\\
&U^{1,0} - U_h^{1,0} = U^{1,0} - I_hU^{1,0} + I_hU^{1,0} - U_h^{1,0} \triangleq \psi^{1,0} + \varphi^{1,0},\\
&u^1 - U_h^{1,0} = u^1 - I_hu^1 + I_hu^1 - U_h^{1,0} \triangleq \eta^1 + \xi^{1,0},\\
&\vec{Q}^{1,0} - \vec{Q}_h^{1,0} = \vec{Q}^{1,0} - \Pi_h\vec{Q}^{1,0} + \Pi_h\vec{Q}^{1,0} - \vec{Q}_h^{1,0} \triangleq \vec{\rho}^{1,0} + \vec{\zeta}^{1,0},\\
&\vec{q}^1 - \vec{Q}_h^{1,0} = \vec{q}^1 - \Pi_h\vec{q}^1 + \Pi_h\vec{q}^1 - \vec{Q}_h^{1,0} \triangleq \vec{r}^1 + \vec{\theta}^{1,0}.
\end{aligned}$$

Theorem 4.2. *Let* $\{u, \vec{q}\}$ *and* $\{U_h^n, \widehat{\vec{Q}}_h^n\}$ *be the solutions of* (2.1) *and* (4.1) *respectively. Assume that* $u \in L^\infty(0,T;H^3(\Omega)\cap H_0^1(\Omega)), u_t \in L^\infty(0,T;H^2(\Omega)), u_{tt} \in L^\infty(0,T;H^2(\Omega)), u_{ttt} \in L^\infty(0,T;L^2(\Omega))$. *Then we have*

$$\begin{aligned}
&\frac{1}{\tau}\|\varphi^{1,0}\|_0^2 + \frac{\mu}{2}\|\nabla\varphi^{1,0}\|_0^2 \le Ch^4 + Ch^2\tau^2,\\
&\frac{1}{\tau}\|\varphi^1\|_0^2 + \|\nabla\varphi^1\|_0^2 \le Ch^4 + Ch^2\tau^2,\\
&\|\varphi^n\|_0^2 + \tau\sum_{i=2}^{n}\|\nabla\widehat{\varphi}^n\|_0^2 \le Ch^4 + Ch^2\tau^2,\\
&\tau\sum_{i=2}^{n}\|\overline{\partial}_t\varphi^i\|_0^2 + \|\nabla\varphi^n\|_0^2 \le Ch^2.
\end{aligned}$$

Proof. For $v_h \in V_h, \vec{w}_h \in \vec{W}_h$, subtracting (4.3) from (2.4) gives

$$\Big(\frac{\varphi^{1,0}-\varphi^0}{\tau}, v_h\Big) - \Big(\frac{\vec{\zeta}^{1,0}+\vec{\zeta}^0}{2}, \nabla v_h\Big) = -\Big(\frac{\psi^{1,0}-\psi^0}{\tau}, v_h\Big) + \Big(\frac{\vec{\rho}^0+\vec{\rho}^{1,0}}{2}, \nabla v_h\Big), \tag{4.6a}$$

$$\begin{aligned}
&\Big(\frac{\vec{\zeta}^{1,0}+\vec{\zeta}^0}{2}, \vec{w}_h\Big) + \Big(a(U_h^0)\frac{\nabla\varphi^0+\nabla\varphi^{1,0}}{2}, \vec{w}_h\Big) \qquad (4.6b)\\
&= -\Big(\frac{\vec{\rho}^0+\vec{\rho}^{1,0}}{2}, \vec{w}_h\Big) - \Big(a(U^0)\frac{\nabla\psi^0+\nabla\psi^{1,0}}{2}, \vec{w}_h\Big) - \Big((a(U^0)-a(U_h^0))\frac{\nabla I_hU^0+\nabla I_hU^{1,0}}{2}, \vec{w}_h\Big).
\end{aligned}$$

Noting $\varphi^0=0, \nabla\varphi^0=0$, taking $v_h=\varphi^{1,0}$, $\overrightarrow{w}_h=\nabla\varphi^{1,0}$ in (4.6), we have

$$\frac{1}{\tau}\|\varphi^{1,0}\|_0^2+\frac{\mu}{2}\|\nabla\varphi^{1,0}\|_0^2\le Ch^4\|\overline{\partial}_tU^{1,0}\|_2^2+C\|\varphi^{1,0}\|_0^2-\Big(a(U^0)\frac{\nabla\psi^0+\nabla\psi^{1,0}}{2},\nabla\varphi^{1,0}\Big)$$
$$-\Big((a(U^0)-a(U_h^0))\frac{\nabla I_hU^0+\nabla I_hU^{1,0}}{2},\nabla\varphi^{1,0}\Big). \tag{4.7}$$

Let $\overline{\varphi}|_K=\frac{1}{|K|}\int_K\varphi dxdy,\ \forall\varphi\in W^{1,\infty}(K)$. Then $\|(\varphi-\overline{\varphi})|_K\|_{0,\infty,K}\le Ch\|\varphi\|_{1,\infty,K}$. This, together with (2.9), gives

$$\begin{aligned}&\Big|\Big(a(U^0)\frac{\nabla\psi^0+\nabla\psi^{1,0}}{2},\nabla\varphi^{1,0}\Big)|\\
\le& C\Big|\Big(\frac{\nabla e^{1,0}-\nabla I_he^{1,0}}{2},\nabla\varphi^{1,0}\Big)\Big|+C\Big|\Big(\frac{\nabla u^1-\nabla I_hu^1+\nabla u^0-\nabla I_hu^0}{2},\nabla\varphi^{1,0}\Big)\Big|\\
\le& Ch^4+Ch^2\tau^2+\frac{\mu}{8}\|\nabla\varphi^{1,0}\|_0^2.\end{aligned} \tag{4.8}$$

By Gagliardo-Nirenberg inequality (see [34]): $\|\psi^0\|_{0,4}\le C|\psi^0|_1^{\frac12}\|\psi^0\|_0^{\frac12}+C\|\psi^0\|_0$, we have

$$\begin{aligned}&\Big|\Big((a(U^0)-a(U_h^0))\frac{\nabla I_hU^0+\nabla I_hU^{1,0}}{2},\nabla\varphi^{1,0}\Big)\Big|\\
=&\Big|\Big((a(U^0)-a(U_h^0))\frac{\nabla\psi^0+\nabla\psi^{1,0}}{2},\nabla\varphi^{1,0}\Big)+\Big((a(U^0)-a(U_h^0))\frac{\nabla e^{1,0}}{2},\nabla\varphi^{1,0}\Big)\\
&+\Big((a(U^0)-a(U_h^0))\frac{\nabla u^0+\nabla u^1}{2},\nabla\varphi^{1,0}\Big)\Big|\\
\le& Ch^4+Ch^2\tau^2+\frac{\mu}{8}\|\nabla\varphi^{1,0}\|_0^2.\end{aligned} \tag{4.9}$$

Inserted into (4.6) these estimates of (4.7)-(4.9) and by (2.9), we obtain

$$\frac{1}{\tau}\|\varphi^{1,0}\|_0^2+\|\nabla\varphi^{1,0}\|_0^2\le Ch^4+Ch^2\tau^2. \tag{4.10}$$

Then from (4.2) and (2.6), we have

$$(a)\ (\overline{\partial}_t\varphi^1,v_h)-(\widehat{\vec{\zeta}}^1,\nabla v_h)=-(\overline{\partial}_t\eta^1,v_h)+(\widehat{\vec{\rho}}^1,\nabla v_h), \tag{4.11a}$$

$$(b)\ (\widehat{\vec{\zeta}}^1,\vec{w}_h)+\Big(a(\frac{U_h^0+U_h^{1,0}}{2})\nabla\widehat{\varphi}^1,\vec{w}_h\Big) \tag{4.11b}$$
$$=-(\widehat{\vec{\rho}}^1,\vec{w}_h)-\Big(a(\frac{U^0+U^{1,0}}{2})\nabla\widehat{\psi}^1,\vec{w}_h\Big)-\Big(a(\frac{U^0+U^{1,0}}{2})-a(\frac{U_h^0+U_h^{1,0}}{2})\nabla I_h\widehat{U}^1,\vec{w}_h\Big).$$

In the same way as above, taking $v_h=\varphi^1$, $\overrightarrow{w}_h=\nabla\varphi^1$ in (4.11)

$$\frac{1}{\tau}\|\varphi^1\|_0^2+\frac{\mu}{2}\|\nabla\varphi^1\|_0^2\le Ch^4\|\overline{\partial}_tU^{1,0}\|_2^2+C\|\varphi^1\|_0^2-\Big(a(\frac{U^0+U^{1,0}}{2})\nabla\widehat{\psi}^1,\nabla\varphi^1\Big)$$
$$-\Big(\Big(a(\frac{U^0+U^{1,0}}{2})-a(\frac{U_h^0+U_h^{1,0}}{2})\Big)\nabla I_h\widehat{U}^1,\nabla\varphi^1\Big). \tag{4.12}$$

Just as (4.8), we have

$$\begin{aligned}&\Big|\Big(a\Big(\frac{U^0+U^{1,0}}{2}\Big)\nabla\widehat{\psi}^1,\nabla\varphi^1\Big)\Big|\\ =&\Big|\Big(\Big(a(\frac{U^0+U^{1,0}}{2})-a(\frac{u^0+u^1}{2})\nabla\widehat{\psi}^1,\nabla\varphi^1\Big)\Big)+\Big(a(\frac{u^0+u^1}{2})\nabla\widehat{\psi}^1,\nabla\varphi^1\Big)\Big|\\ \le&Ch^4+Ch^2\tau^2+\frac{\mu}{8}\|\nabla\varphi^1\|_0^2.\end{aligned}\tag{4.13}$$

For a proper variant, we get

$$\begin{aligned}&\Big|\Big(\Big(a(\frac{U^0+U^{1,0}}{2})-a(\frac{U_h^0+U_h^{1,0}}{2})\Big)\nabla I_h\widehat{U}^1,\nabla\varphi^1\Big)\Big|\\ =&\Big|\Big(\Big(a(\frac{U^0+U^{1,0}}{2})-a(\frac{U_h^0+U_h^{1,0}}{2})\Big)(\nabla\widehat{\psi}^1+\nabla\widehat{e}^1+\nabla\widehat{u}^1),\nabla\varphi^1\Big)\Big|\\ \le&Ch\|\widehat{U}^1\|_2(\|\varphi^{1,0}\|_{0,4}+\|e^{1,0}-I_he^{1,0}\|_{0,4}+\|u^1-I_hu^1\|_{0,4}\\ &\quad+\|\psi^0\|_{0,4})\|\nabla\varphi^1\|_{0,4}+C\Big(\|\varphi^{1,0}\|_{0,4}+\|\psi^{1,0}\|_{0,4}+\|\psi^0\|_{0,4}\Big)\|\nabla\widehat{e}^1\|_{0,4}\|\nabla\varphi^1\|_{0,4}\\ &\quad+C\Big(\|\varphi^{1,0}\|_0+\|\psi^{1,0}\|_4+\|\psi^0\|_4\Big)\|\nabla\widehat{e}^1\|_0\|\nabla\varphi^1\|_0\\ \le&Ch^4+Ch^2\tau^2+\frac{\mu}{8}\|\nabla\varphi^1\|_0^2.\end{aligned}\tag{4.14}$$

Then combining (4.12)-(4.14) with (4.10), implies that

$$\frac{1}{\tau}\|\varphi^1\|_0^2+\|\nabla\varphi^1\|_0^2\le Ch^4+Ch^2\tau^2.$$

For $n\ge 2$, we subtracte (4.1) from (2.3) gives

$$(a)\ (\overline{\partial}_t\varphi^n,v_h)-(\widehat{\vec{\zeta}}^n,\nabla v_h)=-(\overline{\partial}_t\psi^n,v_h)+(\widehat{\vec{\rho}}^n,\nabla v_h),\tag{4.15a}$$

$$\begin{aligned}(b)\ &(\widehat{\vec{\zeta}}^n,\vec{w}_h)+(a(\overline{U}_h^n)\nabla\widehat{\varphi}^n,\vec{w}_h)\\ &=-(\widehat{\vec{\rho}}^n,\vec{w}_h)-(a(\overline{U}^n)\nabla\widehat{\psi}^n,\vec{w}_h)-((a(\overline{U}^n)-a(\overline{U}_h^n)\nabla\widehat{I_hU}^n,\vec{w}_h).\end{aligned}\tag{4.15b}$$

On one hand, taking $v_h=\widehat{\varphi}^n$, $\vec{w}_h=\nabla\widehat{\varphi}^n$ in (4.15), we have

$$\begin{aligned}&\frac{\|\varphi^n\|_0^2-\|\varphi^{n-1}\|_0^2}{2\tau}+\mu\|\nabla\widehat{\varphi}^n\|_0^2\\ \le&Ch^4\|\overline{\partial}_tU^n\|_2^2+C\|\widehat{\varphi}^n\|_0^2-(a(\overline{U}^n)\nabla\widehat{\psi}^n,\nabla\widehat{\varphi}^n)-((a(\overline{U}^n)-a(\overline{U}_h^n))\nabla I_h\widehat{U}^n,\nabla\widehat{\varphi}^n).\end{aligned}\tag{4.16}$$

Similar to (4.13)-(4.14), we have

$$\begin{aligned}&\Big|\Big(a(\overline{U}^n)\nabla\widehat{\psi}^n,\nabla\widehat{\varphi}^n\Big)+\Big((a(\overline{U}^n)-a(\overline{U}_h^n))\nabla I_h\widehat{U}^n,\nabla\widehat{\varphi}^n\Big)\Big|\\ \le&Ch\|\widehat{U}^n\|_2(\|\overline{\varphi}^n\|_{0,4}+\|\overline{e}^n-I_h\overline{e}^n\|_{0,4}+\|\overline{u}^n-I_h\overline{u}^n\|_{0,4})\|\nabla\widehat{\varphi}^n\|_{0,4}\\ &\quad+C\Big(\|\overline{\varphi}^n\|_{0,4}+\|\overline{\psi}^n\|_{0,4}\Big)\|\nabla\overline{e}^n\|_{0,4}\|\nabla\widehat{\varphi}^n\|_0+C\Big(\|\overline{\varphi}^n\|_0+\|\overline{\psi}^n\|_0\Big)\|\nabla\widehat{\varphi}^n\|_0\\ \le&Ch^4+Ch^2\tau^2+C\tau^2\|\nabla\widehat{\varphi}^n\|_0^2+\frac{\mu}{4}\|\nabla\widehat{\varphi}^n\|_0^2.\end{aligned}\tag{4.17}$$

Thus

$$
\begin{aligned}
&\|\varphi^n\|_0^2+\tau\sum_{i=2}^{n}\|\nabla\widehat{\varphi}^n\|_0^2\\
\leq&Ch^4+Ch^2\tau^2+\|\varphi^1\|_0^2+C\tau\sum_{i=1}^{n}\|\varphi^i\|_0^2+C\tau^2\sum_{j=2}^{n}\sum_{i=2}^{j-1}\|\nabla\widehat{\varphi}^i\|_0^2\\
\leq&Ch^4+Ch^2\tau^2. \qquad (4.18)
\end{aligned}
$$

By Gronwall's inequality, we have

$$
\|\varphi^n\|_0^2+\tau\sum_{i=2}^{n}\|\nabla\widehat{\varphi}^n\|_0^2\leq Ch^4+Ch^2\tau^2. \qquad (4.19)
$$

On the other hand, taking $v_h=\overline{\partial}_t\varphi^n$, $\overrightarrow{w}_h=\overline{\partial}_t\nabla\varphi^n$ in (4.15) yields

$$
\begin{aligned}
&\|\overline{\partial}_t\varphi^n\|_0^2+\frac{1}{2\tau}\Big(\|a^{\frac12}(\widehat{U}^n)\nabla\varphi^n\|_0^2-\|a^{\frac12}(\widehat{U}^{n-1})\nabla\varphi^{n-1}\|_0^2\Big)\\
=&-(\overline{\partial}_t\psi^n,\overline{\partial}_t\varphi^n)-\Big(a(\overline{U}^n)\nabla\widehat{\psi}^n,\overline{\partial}_t\nabla\varphi^n\Big)-\Big((a(\overline{U}^n)-a(\overline{U}_h^n))\nabla\widehat{U}_h^n,\overline{\partial}_t\nabla\varphi^n\Big)\\
&+\Big(\frac{a(\overline{U}^n)-a(\overline{U}^{n-1})}{\tau}\nabla\varphi^{n-1},\nabla\varphi^{n-1}\Big)\triangleq\sum_{i=1}^{4}A_i. \qquad (4.20)
\end{aligned}
$$

Obviously

$$
|A_1+A_4|\leq Ch^4\|\overline{\partial}_tU^n\|_2^2+\frac14\|\overline{\partial}_t\varphi^n\|_0^2+C\|\overline{\partial}_t\overline{U}^n\|_{0,\infty}\|\nabla\varphi^{n-1}\|_0^2.
$$

Similar to (4.17), we get

$$
\begin{aligned}
|A_2|&\leq Ch\|\widehat{U}^n\|_2\|\overline{e}^n\|_{0,\infty}\|\overline{\partial}_t\nabla\varphi^n\|_0+Ch^2\|\widehat{U}^n\|_2\|\overline{\partial}_t\nabla\varphi^n\|_0+Ch^2+C\|\nabla\varphi^n\|_0^2+C\|\nabla\varphi^{n-1}\|_0^2\\
&\leq Ch^2+C\|\nabla\varphi^n\|_0^2+C\|\nabla\varphi^{n-1}\|_0^2+\frac14\|\overline{\partial}_t\varphi^n\|_0^2, \qquad (4.21)
\end{aligned}
$$

as well as

$$
\begin{aligned}
|A_3|\leq&Ch\|\widehat{U}^n\|_2\Big(\|\overline{\varphi}\|_{0,4}+\|\overline{\psi}\|_{0,4}\Big)\|\overline{\partial}_t\nabla\varphi^n\|_{0,4}+C\tau\Big(\|\overline{\varphi}\|_{0,4}+\|\overline{\psi}\|_{0,4}\Big)\|\overline{\partial}_t\nabla\varphi^n\|_0\\
&+C\Big(\|\overline{\varphi}\|_0+\|\overline{\psi}\|_0\Big)\|\overline{\partial}_t\nabla\varphi^n\|_0\\
\leq&Ch^2+C\|\nabla\varphi^n\|_0^2+C\|\nabla\varphi^{n-1}\|_0^2+\frac14\|\overline{\partial}_t\varphi^n\|_0^2. \qquad (4.22)
\end{aligned}
$$

Consequently, we obtain

$$
\tau\sum_{i=2}^{n}\|\overline{\partial}_t\varphi^i\|_0^2+\|\nabla\varphi^n\|_0^2\leq Ch^2+\|\nabla\varphi^1\|_0^2+C\tau\sum_{i=1}^{n}\|\nabla\varphi^i\|_0^2\leq Ch^2. \qquad (4.23)
$$

Using Gronwall's inequality again, it follows that

$$
\tau\sum_{i=2}^{n}\|\overline{\partial}_t\varphi^i\|_0^2+\|\nabla\varphi^n\|_0^2\leq Ch^2. \qquad (4.24)
$$

Thus the proof is complete. □

Remark 4.1. Theorem 4.2 serves as a bridge for the following τ-independent superclose analysis. Otherwise, we will need estimate (1.2), which results in certain time-step condition.

Now we are ready to state the main result of this paper.

Theorem 4.3. *Let* $\{u, \vec{q}\}$ *and* $\{U_h^n, \widehat{\vec{Q}}_h^n\}$ *be the solutions of* (2.1) *and* (4.1)-(4.3) *respectively. Under the assumption of Theorem* 4.2, *we have*

$$\|\xi^n\|_1 + \|\widehat{\vec{\theta}}^n\|_0 = O(h^2+\tau^2).$$

Proof. By (4.2) and (2.7), we have

$$(a)\ (\overline{\partial}_t\xi^1, v_h) - (\widehat{\vec{\theta}}^1, \nabla v_h) = -(\overline{\partial}_t\eta^1, v_h) + (\widehat{\vec{r}}^1, \nabla v_h) + (R_1^1, v_h) - (R_2^1, \nabla v_h), \tag{4.25a}$$

$$(b)\ (\widehat{\vec{\theta}}^1, \vec{w}_h) + \Big(a(\frac{U_h^{1,0}+U_h^0}{2})\nabla\widehat{\xi}^1, \vec{w}_h) = -(\widehat{\vec{r}}^1, \vec{w}_h\Big) - (a(\widehat{u}^1)\nabla\widehat{\eta}^1, \vec{w}_h)$$
$$- \Big((a(\widehat{u}^1) - a(\frac{U_h^{1,0}+U_h^0}{2})\nabla I_h\widehat{u}^1, \vec{w}_h)\Big) + (R_2^1, \nabla v_h) + (R_3^1, \nabla v_h). \tag{4.25b}$$

Taking $v_h = \frac{\xi^1}{\tau}, \overrightarrow{w}_h = \frac{\nabla\xi^1}{\tau}$ in (4.25) gives

$$\|\frac{\xi^1}{\tau}\|_0^2 + \frac{\mu}{2\tau}\|\nabla\xi^1\|_0^2 \le Ch^4\|\overline{\partial}_t u^1\|_2^2 + C\frac{h^4+\tau^4}{\tau}\|\widehat{u}^1\|_3^2$$
$$+ \frac{\mu}{8\tau}\|\nabla\xi^1\|_0^2 + \frac{1}{2}\|\frac{\xi^1}{\tau}\|_0^2 - \Big((a(\widehat{u}^1) - a(\frac{U_h^{1,0}+U_h^0}{2}))\nabla I_h\widehat{u}^1, \frac{\nabla\xi^1}{\tau}\Big), \tag{4.26}$$

and

$$\Big|\Big((a(\widehat{u}^1) - a(\frac{U_h^{1,0}+U_h^0}{2}))\nabla I_h\widehat{u}^1, \frac{\nabla\xi^1}{\tau}\Big)\Big|$$
$$= \Big|\Big((a(\widehat{u}^1) - a(\frac{U_h^{1,0}+U_h^0}{2}))(\nabla I_h\widehat{u}^1 - \widehat{u}^1), \frac{\nabla\xi^1}{\tau}\Big) + \Big((a(\widehat{u}^1) - a(\frac{U_h^{1,0}+U_h^0}{2}))\nabla\widehat{u}^1, \frac{\nabla\xi^1}{\tau}\Big)\Big|$$
$$\le Ch\|\widehat{u}^1\|_{2,4}\Big(\|\eta^0\|_0 + \|e^{1,0}\|_0 + \|\psi^{1,0}\|_0\Big)\frac{1}{\tau}\|\nabla\xi^1\|_{0,4}$$
$$+ C\Big(\|\eta^0\|_0 + \|e^{1,0}\|_0 + \|\psi^{1,0}\|_0 + \|\varphi^{1,0}\|_0\Big)\frac{1}{\tau}\|\nabla\xi^1\|_0$$
$$\le C\frac{h^4+\tau^4}{\tau} + \frac{\mu}{8\tau}\|\nabla\xi^1\|_0^2. \tag{4.27}$$

By (2.9), we obtain

$$\tau\|\frac{\xi^1}{\tau}\|_0^2 + \|\xi^1\|_1^2 \le C(h^4+\tau^4). \tag{4.28}$$

From (2.2) to (4.1), it yields that

$$(\overline{\partial}_t\xi^n, v_h) - (\widehat{\vec{\theta}}^n, \nabla v_h) = -(\overline{\partial}_t\eta^n, v_h) + (\widehat{\vec{r}}^n, \nabla v_h) + (R_1^n, v_h) - (R_2^n, \nabla v_h), \tag{4.29a}$$

$$(\widehat{\vec{\theta}}^n, \vec{w}_h) + (a(\overline{U}_h^n)\nabla\widehat{\xi}^n, \vec{w}_h) \tag{4.29b}$$
$$= -(\widehat{\vec{r}}^n, \vec{w}_h) - (a(\overline{u}^n)\nabla\widehat{\eta}^n, \vec{w}_h) - ((a(\overline{u}^n) - a(\overline{U}_h^n))\nabla I_h\widehat{u}^n, \vec{w}_h) + (R_2^n, \vec{w}_h) + (R_3^n, \vec{w}_h).$$

Taking $v_h = \overline{\partial}_t\xi^n$, $\overrightarrow{w}_h = \overline{\partial}_t(\nabla\xi^n)$ in (4.29), we have

$$\begin{aligned}
&(\overline{\partial}_t\xi^n,\overline{\partial}_t\xi^n)+(a(\overline{U}_h^n)\nabla\widehat{\xi}^n,\overline{\partial}_t(\nabla\xi^n)) = -(\overline{\partial}_t\eta^n,\overline{\partial}_t\xi^n)-(a(\overline{u}^n)\nabla\widehat{\eta}^n,\overline{\partial}_t(\nabla\xi^n))\\
&\quad+(R_1^n,\overline{\partial}_t\xi^n)+(R_3^n,\overline{\partial}_t(\nabla\xi^n))-((a(\overline{u}^n)-a(\overline{U}_h^n))\nabla I_h\widehat{u}^n,\overline{\partial}_t(\nabla\xi^n))\\
=&-(\overline{\partial}_t\eta^n,\overline{\partial}_t\xi^n)+(\overline{\partial}_t(a(\overline{u}^n)\nabla\widehat{\eta}^n),\nabla\xi^{n-1})-\overline{\partial}_t(a(\overline{u}^n)\nabla\widehat{\eta}^n,\nabla\xi^n)+(R_1^n,\overline{\partial}_t\xi^n)\\
&\quad-(\overline{\partial}_t(R_3^n),\nabla\xi^{n-1})+\overline{\partial}_t(R_3^n,\nabla\xi^n)-(\overline{\partial}_t((a(\overline{u}^n)-a(\overline{U}_h^n))\nabla I_h\widehat{u}^n),\nabla\xi^{n-1})\\
&\quad+\overline{\partial}_t((a(\overline{u}^n)-a(\overline{U}_h^n))\nabla I_h\widehat{u}^n,\nabla\xi^n)\triangleq\sum_{i=1}^{8}J_i. \qquad (4.30)
\end{aligned}$$

Then the left side of (4.30) can be estimated as

$$\begin{aligned}
&(\overline{\partial}_t\xi^n,\overline{\partial}_t\xi^n)+(a(\overline{U}_h^n)\nabla\widehat{\xi}^n,\overline{\partial}_t(\nabla\xi^n)) \qquad (4.31)\\
=&\|\overline{\partial}_t\xi^n\|_0^2+\frac{\|a^{\frac{1}{2}}(\overline{U}_h^n)\nabla\xi^n\|_0^2-\|a^{\frac{1}{2}}(\overline{U}_h^{n-1})\nabla\xi^{n-1}\|_0^2}{2\tau}-(\frac{a(\overline{U}_h^n)-a(\overline{U}_h^{n-1})}{2\tau}\nabla\xi^{n-1},\nabla\xi^{n-1}).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&|J_7|\le C|((\overline{\partial}_t\overline{\psi}^n+\overline{\partial}_t\overline{\varphi}^n+\overline{\partial}_t\overline{U}^n)\nabla\xi^{n-1},\nabla\xi^{n-1})|\le Ch^{-1}\|\overline{\partial}_t\overline{\varphi}^n\|_0\|\nabla\xi^{n-1}\|_0^2+C\|\nabla\xi^{n-1}\|_0^2,\\
&|J_1+J_4|\le Ch^4\|u_t\|_{L^\infty(H^2)}^2+C\tau^4+\frac{1}{2}\|\overline{\partial}_t\xi^n\|_0^2.
\end{aligned}$$

By use of the mean-value technique, we deduce that

$$\begin{aligned}
|J_2|=&\left|\left(\frac{a(\overline{u}^n)(\nabla\widehat{\eta}^n-\nabla\widehat{\eta}^{n-1})+(a(\overline{u}^n)-a(\overline{u}^{n-1}))\nabla\widehat{\eta}^{n-1}}{\tau},\nabla\xi^{n-1}\right)\right|\\
\le& Ch^4|\overline{\partial}_t\widehat{u}^n|_3^2+Ch^4|\widehat{u}^{n-1}|_3^2+C\|\nabla\xi^{n-1}\|_0^2. \qquad (4.32)
\end{aligned}$$

It is not difficult to see that

$$\begin{aligned}
&\frac{a(\overline{u}^n)-a(\overline{U}_h^n)-(a(\overline{u}^{n-1})-a(\overline{U}_h^{n-1}))}{\tau}\\
=&\frac{a_u(\overline{u}^{n-1})(\overline{u}^n-\overline{u}^{n-1})+\frac{1}{2}a_{uu}(\nu_1^n)(\overline{u}^n-\overline{u}^{n-1})^2}{\tau}\\
&\quad-\frac{a_u(\overline{U}_h^{n-1})(\overline{U}_h^n-\overline{U}_h^{n-1})+\frac{1}{2}a_{uu}(\nu_2^n)(\overline{U}_h^n-\overline{U}_h^{n-1})^2}{\tau}\\
=&\frac{a_u(\overline{U}_h^{n-1})(\overline{U}^n-\overline{U}^{n-1}-\overline{U}_h^n+\overline{U}_h^{n-1})}{\tau}+\frac{(a_u(\overline{u}^{n-1})-a_u(\overline{U}_h^{n-1}))(\overline{u}^n-\overline{u}^{n-1})}{\tau}\\
&\quad+\frac{1}{2}a_{uu}(\nu_2^n)(\overline{u}^n-\overline{U}_h^n-(\overline{u}^{n-1}-\overline{U}_h^{n-1}))(\overline{\partial}_t\overline{u}^n+\overline{\partial}_t\overline{U}_h^n)+\frac{\tau}{2}(a_{uu}(\nu_1^n)-a_{uu}(\nu_2^n))(\overline{\partial}_t\overline{u}^n)^2,
\end{aligned}$$

where $\nu_1^n=\overline{u}^{n-1}+\lambda_1(\overline{u}^n-\overline{u}^{n-1}),\nu_2^n=\overline{U}_h^{n-1}+\lambda_2(\overline{U}_h^n-\overline{U}_h^{n-1}),0<\lambda_1,\lambda_2<1$. Since

$$\begin{aligned}
&\|a_{uu}(\nu_1^n)-a_{uu}(\nu_2^n)\|_0^2\le C\|\nu_1^n-\nu_2^n\|_0^2\\
\le& C\|\overline{u}^{n-1}-\overline{U}_h^{n-1}\|_0^2+C\|\lambda_2^n(\overline{u}^n-\overline{u}^{n-1}-\overline{U}_h^n+\overline{U}_h^{n-1})\|_0^2+C\|(\overline{u}^n-\overline{u}^{n-1})(\lambda_1-\lambda_2)\|_0^2\\
\le& C\|\widehat{\xi}^n\|_0^2+C\|\widehat{\xi}^{n-1}\|_0^2+Ch^4+C\tau^2, \qquad (4.33)
\end{aligned}$$

we have

$$
\begin{aligned}
|J_6| =& \left|\left(\frac{((a(\overline{u}^n)-a(\overline{U}_h^n))-(a(\overline{u}^{n-1})-a(\overline{U}_h^{n-1})))\nabla I_h\widehat{u}^n}{\tau},\nabla\xi^{n-1}\right)\right| \\
&+\left|\left(\frac{(a(\overline{u}^{n-1})-a(\overline{U}_h^{n-1}))(\nabla I_h\widehat{u}^n-\nabla I_h\widehat{u}^{n-1})}{\tau},\nabla\xi^{n-1}\right)\right| \\
\le& Ch^4+C\tau^4+\frac{1}{4}\|\overline{\partial}_t\overline{\xi}^{n-1}\|_0^2+C\|\nabla\xi^{n-1}\|_0^2+C\|\nabla\xi^{n-2}\|_0^2. \qquad (4.34)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\frac{R_3^n-R_3^{n-1}}{\tau} =& \left(a(\overline{u}^{n-1})-a(u^{n-1-\frac{1}{2}})\right)\frac{\nabla\widehat{u}^n-\nabla\widehat{u}^{n-1}}{\tau} \\
&+\nabla\widehat{u}^n\frac{(a(\overline{u}^n)-a(u^{n-\frac{1}{2}}))-(a(\overline{u}^{n-1})-a(u^{n-1-\frac{1}{2}}))}{\tau} \\
&+(\nabla\widehat{u}^{n-1}-\nabla u^{n-1-\frac{1}{2}})\frac{a(u^{n-\frac{1}{2}})-a(u^{n-1-\frac{1}{2}})}{\tau} \\
&+a(u^{n-\frac{1}{2}})\frac{(\nabla\widehat{u}^n-\nabla u^{n-\frac{1}{2}})-(\nabla\widehat{u}^{n-1}-\nabla u^{n-1-\frac{1}{2}})}{\tau},
\end{aligned}
$$

we have

$$
|J_5|\le C\tau^4+C\|\nabla\xi^{n-1}\|_0^2.
$$

Inserting the estimates of J_1-J_8 into (4.30), we have

$$
\begin{aligned}
&\|\overline{\partial}_t\xi^n\|_0^2+\frac{\|a^{\frac{1}{2}}(\overline{U}_h^n)\nabla\xi^n\|_0^2-\|a^{\frac{1}{2}}(\overline{U}_h^{n-1})\nabla\xi^{n-1}\|_0^2}{2\tau} \\
\le& Ch^4+C\tau^4+Ch^{-1}\|\overline{\partial}_t\overline{\varphi}^n\|_0\|\nabla\xi^{n-1}\|_0^2+\overline{\partial}_t(R_3^n,\nabla\xi^n)-\overline{\partial}_t(a(\overline{u}^n)\nabla\widehat{\eta}^n,\nabla\xi^n) \\
&+\overline{\partial}_t((a(\overline{u}^n)-a(\overline{U}_h^n))\nabla I_h\widehat{u}^n,\nabla\xi^n)+C\|\nabla\xi^n\|^2+C\|\nabla\xi^{n-1}\|^2+C\|\nabla\xi^{n-2}\|^2.
\end{aligned}
$$

Summing up from 2 to $n<N$ yields

$$
\tau\sum_{i=2}^n\|\overline{\partial}_t\xi^i\|_0^2+\|\nabla\xi^n\|_0^2\le Ch^4+C\tau^4+Ch^{-1}\tau\sum_{i=2}^n\|\overline{\partial}_t\overline{\varphi}^i\|_0\|\nabla\xi^{i-1}\|_0^2+C\tau\sum_{i=1}^n\|\nabla\xi^i\|_0^2.
$$

Note that

$$
Ch^{-1}\tau\sum_{i=2}^n\|\overline{\partial}_t\overline{\varphi}^i\|_0\le Ch^{-1}\tau^{\frac{1}{2}}\left(\sum_{i=2}^n\|\overline{\partial}_t\overline{\varphi}^i\|_0^2\right)^{\frac{1}{2}}\le C.
$$

This, together with (4.24) and the Gronwall's inequality, gives

$$
\tau\sum_{i=2}^n\|\overline{\partial}_t\xi^i\|_0^2+\|\nabla\xi^n\|_0^2\le Ch^4+C\tau^4,
$$

which implies the result

$$
\|\xi^n\|_1=O(h^2+\tau^2). \qquad (4.35)
$$

With the estimates of (3.1) and (3.2), we take $\vec{w}_h=\widehat{\vec{\theta}}^n$ in (4.29) to get

$$
\|\widehat{\vec{\theta}}^n\|_0=O(h^2+\tau^2). \qquad (4.36)
$$

Then the proof is complete. □

Remark 4.2. We point out that the unconditional superclose results of Theorem 4.3 improve the corresponding results of [25-30]. At the same time, Theorem 4.3 is also valid to some other known finite element pairs, which satisfy the special properties (3.1) and (3.2). For example, the conforming element pair $P_1/P_0 \times P_0$ (on triangular meshes), the nonconforming pairs $Q_1^{rot}/Q_{10} \times Q_{01}$ (on square meshes), $EQ_1^{rot}/Q_{10} \times Q_{01}$ (on rectangular meshes) and so on. In this case, the above two nonconforming element pairs satisfy

$$(\nabla(I_h u - u), \nabla v) = 0, \qquad \forall v \in V_h, \tag{4.37}$$

$$\Big| \sum_{K\in T_h} \int_{\partial K} \vec{q} \cdot \vec{n} v_h ds \Big| \le Ch^2 \|\vec{q}\|_2 \|v_h\|_h, \qquad \forall v_h \in V_h, \tag{4.38}$$

where $\|\cdot\|_h = \big(\sum_{K\in T_h} |\cdot|^2_{1,K}\big)^{\frac{1}{2}}$, $\vec{n}$ denotes the outward unit normal vector to ∂K. Thus our analysis can be regarded as a framework to deal with unconditional superclose estimates of the low order mixed FEMs.

5. Numerical Results

We consider problem (1.1) with $\Omega = [0,1] \times [0,1]$, $T = 1$, $a(u) = \sin(u) + 0.1$. We choose f so that the exact solution to (1.1) is $u = e^t xy(1-x)(1-y)$. Then, we have $\vec{q} = -a(u)\nabla u = (\vec{q_1}, \vec{q_2}) = (-(\sin(e^t xy(1-x)(1-y)) + 0.1)e^t y(1-y)(1-2x), -(\sin(e^t xy(1-x)(1-y)) + 0.1)e^t x(1-x)(1-2y))$.

Table 5.1: Numerical results of u at $t = 0.5$.

$m \times m$	$\|u^n - U_h^n\|_1$	Order	$\|I_h u^n - U_h^n\|_1$	Order
4×4	0.0618	–	0.0136	–
8×8	0.0308	1.0060	0.0035	1.9474
16×16	0.0154	1.0015	0.0009	1.9853
32×32	0.0077	1.0000	0.0002	1.9956

Table 5.2: Numerical results of u at $t = 0.75$.

$m \times m$	$\|u^n - U_1^n\|_h$	Order	$\|I_h u^n - U_h^n\|_1$	Order
4×4	0.0794	–	0.0188	–
8×8	0.0395	1.0063	0.0049	1.9352
16×16	0.0197	1.0016	0.0012	1.9814
32×32	0.0098	1.0073	0.0003	2.0000

Table 5.3: Numerical results of u at $t = 1$.

$m \times m$	$\|u^n - U_h^n\|_1$	Order	$\|I_h u^n - U_h^n\|_1$	Order
4×4	0.1019	–	0.0242	–
8×8	0.0507	1.0063	0.0064	1.9272
16×16	0.0253	1.0017	0.0016	1.9783
32×32	0.0126	1.0057	0.0004	2.0000

Table 5.4: Numerical results of $\vec{q}$ at $t = 0.5$.

$m \times m$	$\|\vec{q}^n - \vec{Q}_h^n\|_0$	Order	$\|\Pi_h \vec{q}^n - \vec{Q}_h^n\|_0$	Order
4×4	0.0077	–	1.3306×10^{-3}	–
8×8	0.0039	0.9783	0.3310×10^{-3}	2.0070
16×16	0.0020	0.9936	0.0827×10^{-3}	2.0014
32×32	0.0010	1.0000	0.0207×10^{-3}	1.9983

Table 5.5: Numerical results of $\vec{q}$ at $t = 0.75$.

$m \times m$	$\|\vec{q}^n - \vec{Q}_h^n\|_0$	Order	$\|\Pi_h \vec{q}^n - \vec{Q}_h^n\|_0$	Order
4×4	0.0109	–	1.7471×10^{-3}	–
8×8	0.0056	0.9556	0.4350×10^{-3}	2.0059
16×16	0.0028	0.9879	0.1086×10^{-3}	2.0014
32×32	0.0014	1.0000	0.0272×10^{-3}	1.9973

Table 5.6: Numerical results of $\vec{q}$ at $t = 1$.

$m \times m$	$\|\vec{q}^n - \vec{Q}_h^n\|_0$	Order	$\|\Pi_h \vec{q}^n - \vec{Q}_h^n\|_0$	Order
4×4	0.0157	–	2.1713×10^{-3}	–
8×8	0.0083	0.9285	0.5391×10^{-3}	2.0099
16×16	0.0042	0.9814	0.1345×10^{-3}	2.0029
32×32	0.0021	1.0000	0.0336×10^{-3}	2.0054

Table 5.7: Errors of u in $\|\cdot\|_1$ and errors of $\vec{q}$ in $\|\cdot\|_0$ with $\tau = kh$.

$\|\vec{q}^n - \vec{Q}_h^n\|_0$				$\|u^n - U_h^n\|_1$		
k=1	k=4	k=8	t	k=1	k=4	k=8
0.0010	0.0010	0.0010	0.5	0.0077	0.0077	0.0077
0.0014	0.0014	0.0015	0.75	0.0099	0.0099	0.0099
0.0021	0.0021	0.0025	1	0.0127	0.0127	0.0127

In our computation, a uniform rectangular partition with m+1 nodes in each direction is used and $h = 5\tau$ is chosen for the linearized FEM. It can be seen from Tables 5.1-5.6 that $\|u^n - U_h^n\|_h$ and $\|\vec{q}^n - \vec{Q}_h^n\|_0$ are convergent at rate of $O(h)$, $\|I_h u^n - U_h^n\|_1$ and $\|\Pi_h \vec{q}^n - \vec{Q}_h^n\|_0$ are convergent at rate of $O(h^2)$, which coincide with our theoretical analysis.

To reveal the unconditional stability, we solve the system with a fixed $h = \frac{1}{32}$ and several different time steps $\tau = h, 4h, 8h$. From the numerical results listed in Table 5.7, we can observe that the errors of $\|u^n - U_h^n\|_1$ and $\|\vec{q}^n - \vec{Q}_h^n\|_0$ tend to be a constant as $\frac{h}{\tau} \to 0$, respectively, which implies that the time-restrictions are not necessary.

At the same time, we describe the error reduction results in Figs. 5.1-5.6, where J_1^u and J_2^u denote $\|u^n - U_h^n\|_1$ and $\|I_h u^n - U_h^n\|_1$, respectively, and J_1^q and J_2^q denote $\|\vec{q}^n - \vec{Q}_h^n\|_0$ and $\|\Pi_h \vec{q}^n - \vec{Q}_h^n\|_0$, respectively.

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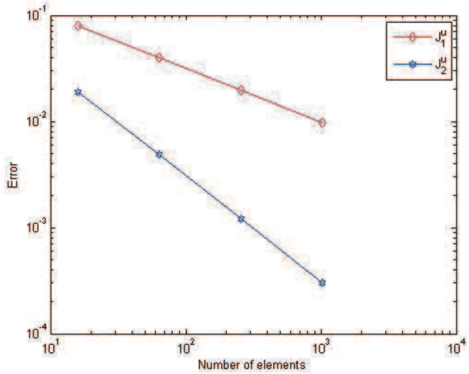

Fig. 5.1: Error reduction results for u at $t = 0.5$.

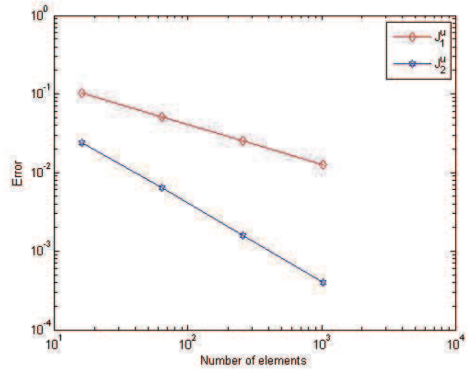

Fig. 5.2: Error reduction results for $\vec{q}$ at $t = 0.5$.

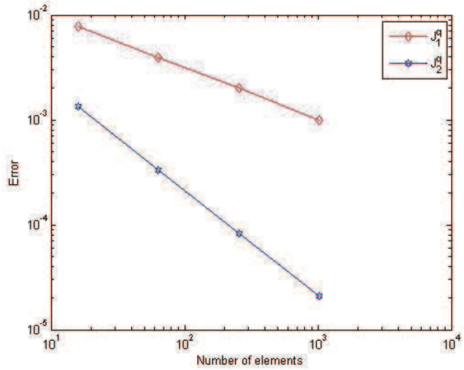

Fig. 5.3: Error reduction results for u at $t = 0.75$.

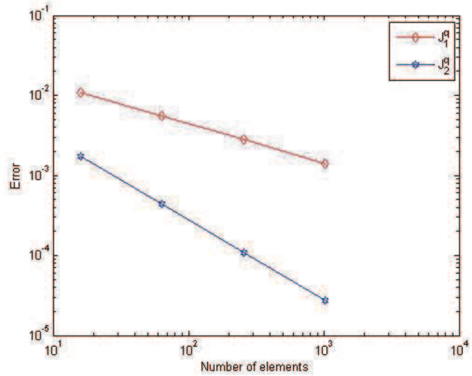

Fig. 5.4: Error reduction results for $\vec{q}$ at $t = 0.75$.

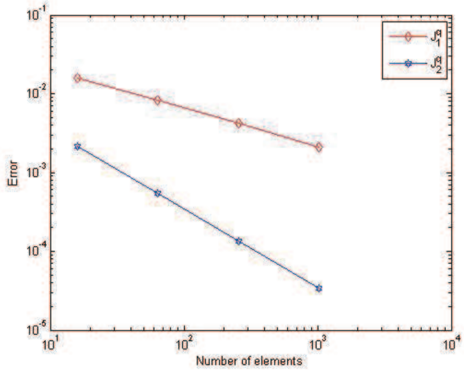

Fig. 5.5: Error reduction results for u at $t = 1$.

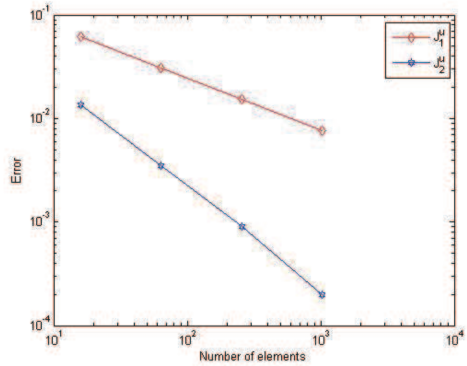

Fig. 5.6: Error reduction results for $\vec{q}$ at $t = 1$.

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