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# UNIFORMLY CONVERGENT NONCONFORMING TETRAHEDRAL ELEMENT FOR DARCY-STOKES PROBLEM*

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**Abstract**

In this paper, we construct a tetrahedral element named DST20 for the three dimensional Darcy-Stokes problem, which reduces the degrees of velocity in [30]. The finite element space $\boldsymbol{V}_h$ for velocity is $\boldsymbol{H}(\mathrm{div})$-conforming, i.e., the normal component of a function in $\boldsymbol{V}_h$ is continuous across the element boundaries, meanwhile the tangential component of a function in $\boldsymbol{V}_h$ is average continuous across the element boundaries, hence $\boldsymbol{V}_h$ is $\boldsymbol{H}^1$-average conforming. We prove that this element is uniformly convergent with respect to the perturbation constant $\varepsilon$ for the Darcy-Stokes problem. At the same time, we give a discrete de Rham complex corresponding to DST20 element.

*Mathematics subject classification:* 65N15, 65N30.
*Key words:* Darcy-Stokes problem, Mixed finite elements, Tetrahedral element, Uniformly convergent.

## 1. Introduction

In this paper, we consider the mixed finite element methods for the following singular perturbation problem of three dimension [12, 30]:

$$\begin{cases} (I-\varepsilon^2\boldsymbol{\Delta})\boldsymbol{u}-\mathbf{grad}p=\boldsymbol{f} & \text{in } \boldsymbol{\Omega}, \\ \mathrm{div}\boldsymbol{u}=0 & \text{in } \Omega, \\ \boldsymbol{u}=0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega\subset\mathbb{R}^3$ is a bounded, convex and connected polygonal domain with boundary $\partial\Omega$, $\varepsilon\in(0,1]$ is a parameter, $\boldsymbol{\Delta}$ is the standard Laplace operator. The vector field $\boldsymbol{u}$ and the scalar field $p$ are corresponding to velocity and pressure in flow problems, respectively.

The problem (1.1) admits a unique solution and $p$ is determined only up to addition of a constant [22]. When $\varepsilon$ is not too small, this problem is simply a standard Stokes problem, but with an additional non-harmful lower order term. If $\boldsymbol{f}=0$ and $\varepsilon$ approaches zero, the problem (1.1) tends to a mixed formulation of the Poisson equation with homogeneous Neumann boundary conditions i.e. a Darcy flow. When $\varepsilon=0$, the first equation of (1.1) has the form of Darcy's law for flow in a homogeneous porous medium. Generalizations of the system (1.1) have been proposed in various physical models, see, e.g., [14, 15, 17, 21, 23, 24, 31, 33].

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In order to make the discrete problem using mixed finite element method for (1.1) well posed, one has to be careful to choose the velocity/pressure finite element spaces. One of the usual methods( [6] etc.) chooses nonconforming Crouzeix-Raviart elements [13] that are convergent for Stokes problem, and we also know that Raviart-Thomas elements that are convergent for the mixed two order problem [28], are not uniformly convergent with respect to the perturbation constant $\varepsilon$. Several methods are presented to construct uniformly convergent elements for (1.1). The first method uses $H^1$-conforming elements for velocity but on the special meshes, such as [3,27,29,32,34]. The second method is stabilized method based on different approaches, such as [4,8,9,18–20,31]. The third method uses $H(\text{div},\Omega)$-conforming but $H^1$-nonconforming elements [10,16,22,35].

In three dimension case, the bubble function method proposed in [11] for 3D fourth-order elliptic problem can also be employed in the construction of uniformly convergent finite elements for the Darcy-Stokes problem. Tai & Wither, 2006, [30] presented a $H(\text{div})$-conforming and uniformly convergent tetrahedron element with 24 degrees of freedom for velocity. In this paper, we present a $H(\text{div})$-conforming and uniformly convergent tetrahedron element with 20 degrees of freedom for velocity. We name the element DSC20 element. Another object of this paper is to construct the discrete de Rham complex corresponding to DSC20 element. Discrete de Rham complex are fundamental tools in the construction of stable elements for some finite element methods [1,2]. Well-known examples of such finite element spaces are described in [25,26]. In three space dimensions the Sobolev space version of the de Rham complex can be written in the form

$$R \xrightarrow{\subset} H^2 \xrightarrow{\textbf{grad}} \boldsymbol{H}(\textbf{curl}) \xrightarrow{\textbf{curl}} \boldsymbol{H}(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

A corresponding discrete de Rham complex is the form

$$R \xrightarrow{\subset} S_h \xrightarrow{\textbf{grad}} \boldsymbol{W}_h \xrightarrow{\textbf{curl}} \boldsymbol{V}_h \xrightarrow{\text{div}} Q_h \longrightarrow 0,$$

where $S_h$, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$ and $Q_h$ are conforming or nonconforming finite element spaces of $\boldsymbol{H}^2(\Omega)$, $\boldsymbol{H}(\textbf{curl},\Omega)$, $\boldsymbol{H}(\text{div},\Omega)$ and $L^2(\Omega)$, respectively.

In our discrete de Rham complex, $S_h$ is $H^1$-conforming and $H^2$-average conforming, it is convergent for the fourth order elliptic problem and uniformly convergent for the fourth order singular perturbation problem; $\boldsymbol{W}_h$ is $\boldsymbol{H}(\textbf{curl})$-conforming and $H^1$-average conforming, $\boldsymbol{V}_h$ presented in this paper is $\boldsymbol{H}(\text{div})$-conforming and $H^1$-average conforming, it is uniformly convergent for Darcy-Stokes singular perturbation problem.

The rest of this paper is organized as follows. In section 2, we introduce the notation and some well-known results of the Darcy-Stokes problem presented in [22]. The construction of DSC20 element is given in section 3. In section 4, we discuss the uniform convergence and the uniform error estimates of the discrete Darcy-Stokes problem. The last section, we construct a discrete de Rham complex corresponding to DST20 elements.

## 2. Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a convex and bounded polygon, $H^m(\Omega)$ and $H_0^m(\Omega)$ be the usual Sobolev spaces with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$ respectively, $H^{-m}(\Omega)$ be the dual space of $H_0^m(\Omega)$, $L_0^2(\Omega)$ be the space of $L^2(\Omega)$ functions with mean value zero. Bold-faces are used to denote the vector functions.

The differential operators are defined as the following:

If $q$ is scalar function, then

$$\mathbf{grad}q = \left(\frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2}, \frac{\partial q}{\partial x_3}\right)^{\top}.$$

The gradient of a vector field $\boldsymbol{v}$ is denoted by $\boldsymbol{Dv}$, it is a $3 \times 3$ matrix with elements

$$(\boldsymbol{Dv})_{i,j} = \frac{\partial v_i}{\partial x_j}, \qquad 1 \le i, j \le 3.$$

If $\boldsymbol{u} = (u_1, u_2, u_3)^{\top}$ is a vector function, then $\mathrm{div}\boldsymbol{u} = \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3}$,

$$\mathbf{curl}\boldsymbol{u} = \mathbf{curl} \wedge \boldsymbol{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

The notation $P_k(T)$ means the space of polynomials of degree $k$ defined on $T$, and $P_k^n(T)$ denotes the corresponding space of polynomial vector fields. These definitions lead to the following Green's formula:

$$-\int_\Omega \boldsymbol{\Delta u} \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{Du} : \boldsymbol{Dv}\, dx, \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^2(\Omega), \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega).$$

In addition to the above spaces, we will also use the following spaces:

$$\boldsymbol{H}(\mathrm{div}, \Omega) = \Big\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega); \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\Big\}, \text{ with } ||\boldsymbol{v}||^2_{\mathrm{div},\Omega} = ||\boldsymbol{v}||^2_{0,\Omega} + ||\mathrm{div}\boldsymbol{v}||^2_{0,\Omega},$$
$$\boldsymbol{H}_0(\mathrm{div}, \Omega) = \Big\{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega); \boldsymbol{v} \cdot \boldsymbol{n} = 0, \text{ on } \partial\Omega\Big\}.$$

A weak formulation of problem (1.1) is given: find $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{cases} a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) + (p, \mathrm{div}\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), & \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\ (\mathrm{div}\boldsymbol{u}, q) = 0, & \forall q \in L_0^2(\Omega), \end{cases} \tag{2.1}$$

where $a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}) + \varepsilon^2(\boldsymbol{Du}, \boldsymbol{Dv})$ and $(\alpha, \beta) = \int_\Omega \alpha\beta\, dx$.

The reduced system ($\varepsilon = 0$) corresponding to (1.1) is

$$\begin{cases} \boldsymbol{u}^0 - \mathbf{grad}p^0 = \boldsymbol{f}, & \text{in } \Omega, \\ \mathrm{div}\boldsymbol{u}^0 = 0, & \text{in } \Omega, \\ \boldsymbol{u}^0 \cdot \boldsymbol{n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

This system has a weak formulation given by (2.1) with $\varepsilon = 0$, but the solution space $\boldsymbol{H}_0^1(\Omega)$ replaced by $\boldsymbol{H}_0(\mathrm{div}, \Omega)$. The energy norm $|||\cdot|||_\varepsilon$ is defined by

$$|||\boldsymbol{v}|||^2_\varepsilon = ||\boldsymbol{v}||^2_{0,\Omega} + ||\mathrm{div}\boldsymbol{v}||^2_{0,\Omega} + \varepsilon^2||\boldsymbol{Dv}||^2_{0,\Omega}. \tag{2.3}$$

Considering

$$\boldsymbol{Z} = \Big\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega);\ (q, \mathrm{div}\boldsymbol{v}) = 0,\ \forall q \in \boldsymbol{L}_0^2(\Omega)\Big\} = \Big\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega);\ \mathrm{div}\boldsymbol{v} = 0\Big\},$$

for all $\boldsymbol{v} \in \boldsymbol{Z}$, we get

$$a_\varepsilon(\boldsymbol{v}, \boldsymbol{v}) = (\boldsymbol{v}, \boldsymbol{v}) + \varepsilon^2(\boldsymbol{D}\boldsymbol{v}, \boldsymbol{D}\boldsymbol{v}) = |||\boldsymbol{v}|||_\varepsilon^2.$$

Due to Lemma 11.2.3 of [1], $\forall q \in \boldsymbol{L}_0^2(\Omega), \exists \boldsymbol{v}_0 \in \boldsymbol{H}_0^1(\Omega), \mathrm{div}\boldsymbol{v}_0 = q, ||\boldsymbol{v}_0||_1 \le c||q||_0$, and

$$|||\boldsymbol{v}|||_\varepsilon \le |||\boldsymbol{v}|||_1 \le ||\boldsymbol{v}||_1, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),$$
$$\sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{(q, \mathrm{div}\boldsymbol{v})}{|||\boldsymbol{v}|||_\varepsilon} \ge \frac{(q, \mathrm{div}\boldsymbol{v}_0)}{|||\boldsymbol{v}_0|||_\varepsilon} \ge \frac{(q, q)}{c||q||_0} = \frac{1}{c}||q||_0,$$

so (2.1) has one and only one solution.

**Lemma 2.1.** ([30]) *There exists constant $c$ independent of $\varepsilon$ and $\boldsymbol{f}$, such that*

$$||\boldsymbol{u}^0||_1 + ||p^0||_2 \le c||\boldsymbol{f}||_1. \tag{2.4}$$

**Lemma 2.2.** ([30]) *Assume that $\Omega$ is convex and $\boldsymbol{f} \in \boldsymbol{H}^1(\Omega)$. There exists a constant $c \ge 0$, independent of $\varepsilon$ and $\boldsymbol{f}$, such that*

$$\varepsilon^{\frac{1}{2}}||\boldsymbol{u}||_1 + \varepsilon^{\frac{3}{2}}||\boldsymbol{u}||_2 \le ||\boldsymbol{f}||_1, \tag{2.5}$$
$$||\boldsymbol{u} - \boldsymbol{u}^0||_0 + ||p - p^0||_1 \le c\varepsilon^{\frac{1}{2}}||\boldsymbol{f}||_1. \tag{2.6}$$

## 3. Construction of the Element DSC20

Let the reference element be $\hat{T} = \{\boldsymbol{x};\ x_i \ge 0, 1 \le i \le 3, 0 \le x_1 + x_2 + x_3 \le 1\}$, whose vertices are $\hat{a}_1(1,0,0)$, $\hat{a}_2(0,1,0)$, $\hat{a}_3(0,0,1)$, $\hat{a}_4(0,0,0)$, respectively; let the opposite face be denoted $\hat{f}_i$, $1 \le i \le 4$, $\hat{\boldsymbol{n}}_i$ is a unit normal vector on $\hat{f}_i$.

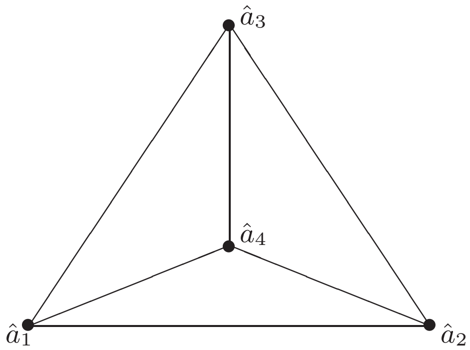


Fig. 3.1. a reference element.

The shape function space of the velocity is defined by

$$\boldsymbol{V}(\hat{T}) = P_1^3(\hat{T}) \oplus \mathbf{curl}(\hat{b}\tilde{P}_1^{*3}), \tag{3.1}$$
$$\tilde{P}_1^{*3} = \{\boldsymbol{v} \in \tilde{P}_1^3; \mathrm{div}\boldsymbol{v} = 0\}, \tag{3.2}$$

where $\hat{b} = \lambda_1\lambda_2\lambda_3\lambda_4, \lambda_i = \hat{x}_i, 1 \le i \le 3, \lambda_4 = 1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3$, $\tilde{P}_1$ is homogeneous polynomial of $P_1$.

**Lemma 3.1.** $\boldsymbol{V}(\hat{T})$ *has the following properties:*

$$\dim \boldsymbol{V}(\hat{T}) = 20, \tag{3.3}$$

$$\operatorname{div} \boldsymbol{V}(\hat{T}) = P_0, \tag{3.4}$$

$$\mathbf{curl}(\hat{b}\tilde{P}_1^{*3}) \cdot \hat{\boldsymbol{n}}\big|_{\hat{f}_i} \equiv 0, \qquad 1 \le i \le 4. \tag{3.5}$$

*Proof.* (1) $\dim \boldsymbol{V}(\hat{T}) = \dim P_1^3 + \dim \tilde{P}_1^{*3} = 4\times 3 - 3\times 3 - 1 = 20$. (2) Because of $\operatorname{div}\mathbf{curl} = 0$, (3.4) is concluded.

(3) In fact, we obtain

$$\mathbf{curl}(\hat{b}\hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{n}}_i\big|_{\hat{f}_i} \equiv 0, \quad 1 \le i \le 4,\ \forall \hat{\boldsymbol{v}} \in \boldsymbol{C}^1(\hat{T}). \tag{3.6}$$

Define $\hat{\boldsymbol{v}} = (v_1, v_2, v_3)$. Then it holds that

$$\mathbf{curl}(\hat{b}\hat{\boldsymbol{v}}) = \left(\frac{\partial(\hat{b}v_3)}{\partial\hat{x}_2} - \frac{\partial(\hat{b}v_2)}{\partial\hat{x}_3}, \frac{\partial(\hat{b}v_1)}{\partial\hat{x}_3} - \frac{\partial(\hat{b}v_3)}{\partial\hat{x}_1}, \frac{\partial(\hat{b}v_2)}{\partial\hat{x}_1} - \frac{\partial(\hat{b}v_1)}{\partial\hat{x}_2}\right)^{\top}, \tag{3.7}$$

on $\hat{f}_1$, $\hat{x}_1 = 0$, $\hat{\boldsymbol{n}}_1 = (1,0,0)$,

$$\mathbf{curl}(\hat{b}\hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{n}}_1\Big|_{\hat{x}_1=0} \overset{(3.7)}{=} \left(\frac{\partial(\hat{b}v_3)}{\partial\hat{x}_2} - \frac{\partial(\hat{b}v_2)}{\partial\hat{x}_3}\right)\Big|_{\hat{x}_1=0} = 0,$$

on $\hat{f}_2$, $\hat{x}_2 = 0$, $\hat{\boldsymbol{n}}_2 = (0,1,0)$,

$$\mathbf{curl}(\hat{b}\hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{n}}_2\Big|_{\hat{x}_2=0} \overset{(3.7)}{=} \left(\frac{\partial(\hat{b}v_1)}{\partial\hat{x}_3} - \frac{\partial(\hat{b}v_3)}{\partial\hat{x}_1}\right)\Big|_{\hat{x}_2=0} = 0,$$

on $\hat{f}_3$, $\hat{x}_3 = 0$, $\hat{\boldsymbol{n}}_3 = (0,0,1)$,

$$\mathbf{curl}(\hat{b}\hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{n}}_3\Big|_{\hat{x}_3=0} \overset{(3.7)}{=} \left(\frac{\partial(\hat{b}v_2)}{\partial\hat{x}_1} - \frac{\partial(\hat{b}v_1)}{\partial\hat{x}_2}\right)\Big|_{\hat{x}_3=0} = 0,$$

on $\hat{f}_4$, $1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3 = 0$, $\hat{\boldsymbol{n}}_4 = \frac{1}{\sqrt{3}}(1,1,1)$,

$$\sqrt{3}\mathbf{curl}(\hat{b}\hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{n}}_4\big|_{\hat{f}_4} \overset{(3.7)}{=} \left[\left(\frac{\partial(\hat{b}v_3)}{\partial\hat{x}_2} - \frac{\partial(\hat{b}v_3)}{\partial\hat{x}_1}\right) + \left(\frac{\partial(\hat{b}v_2)}{\partial\hat{x}_1} - \frac{\partial(\hat{b}v_2)}{\partial\hat{x}_3}\right) + \left(\frac{\partial(\hat{b}v_1)}{\partial\hat{x}_3} - \frac{\partial(\hat{b}v_1)}{\partial\hat{x}_2}\right)\right]\Big|_{\hat{f}_4},$$

$$\left(\frac{\partial(\hat{b}v_3)}{\partial\hat{x}_2} - \frac{\partial(\hat{b}v_3)}{\partial\hat{x}_1}\right)\Big|_{\hat{f}_4} = \left[\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_3}{\partial x_1}\right)\hat{b} + (\hat{x}_1\hat{x}_3 - \hat{x}_2\hat{x}_3)\lambda_4 v_3\right]\Big|_{\hat{f}_4} = p_1\lambda_4\big|_{\lambda_4=0} = 0.$$

Using a similar method, we have

$$\left(\frac{\partial(\hat{b}v_2)}{\partial\hat{x}_1} - \frac{\partial(\hat{b}v_2)}{\partial\hat{x}_3}\right)\Big|_{\hat{f}_4} = p_2\lambda_4\big|_{\lambda_4=0} = 0,$$

$$\left(\frac{\partial(\hat{b}v_1)}{\partial\hat{x}_3} - \frac{\partial(\hat{b}v_1)}{\partial\hat{x}_2}\right)\Big|_{\hat{f}_4} = p_3\lambda_4\big|_{\lambda_4=0} = 0,$$

where $p_1, p_2, p_3$ are polynomial. So (3.6) is complete. □

The degrees of freedom are taken as

$$\int_{\hat{f}_i} \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}} p\, d\hat{s}, \qquad \forall p\in P_1(\hat{f}_i),\quad 1\le i\le 4, \tag{3.8a}$$

$$\int_{\hat{f}_i} \hat{\boldsymbol{v}}\wedge\hat{\boldsymbol{n}}\, d\hat{s}, \qquad 1\le i\le 4. \tag{3.8b}$$

**Lemma 3.2.** *An element* $\hat{\boldsymbol{v}}\in \boldsymbol{V}(\hat{T})$ *is uniquely determined by the degrees of freedom* (3.8).

*Proof.* Because of (3.8) and (3.3), it is enough to show that if $\hat{\boldsymbol{v}}\in \boldsymbol{V}(\hat{T})$ and all the degrees of freedom of $\hat{\boldsymbol{v}}$ are zero, the $\hat{\boldsymbol{v}}=0$. To this end, assume that

$$\int_{\hat{f}_i} \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}} p\, d\hat{s}=0, \qquad \forall p\in P_1(\hat{f}_i),\quad 1\le i\le 4, \tag{3.9a}$$

$$\int_{\hat{f}_i} \hat{\boldsymbol{v}}\wedge\hat{\boldsymbol{n}}\, d\hat{s}=0, \qquad 1\le i\le 4. \tag{3.9b}$$

The definition of $\boldsymbol{V}(\hat{T})$ and (3.5) imply $\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}\big|_{\hat{f}_i}\in P_1(\hat{f}_i)$ and (3.9a). There holds

$$\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}\big|_{\hat{f}_i}\equiv 0, \qquad 1\le i\le 4. \tag{3.10}$$

It follows from (3.4) that

$$\mathrm{div}\hat{\boldsymbol{v}}=\frac{1}{|\hat{T}|}\int_{\hat{T}}\mathrm{div}\hat{\boldsymbol{v}}\, d\hat{x}=\frac{1}{\hat{T}}\int_{\partial\hat{T}}\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}\, d\hat{x}\overset{(3.10)}{=}0. \tag{3.11}$$

Consequently,

$$\hat{\boldsymbol{v}}\in \boldsymbol{H}\overset{\triangle}{=}\Big\{\hat{\boldsymbol{v}}\in \boldsymbol{H}(\mathrm{div},\hat{T});\ \mathrm{div}\boldsymbol{v}=0,\ \boldsymbol{v}\cdot\boldsymbol{n}\big|_{\partial\hat{T}}=0\Big\}. \tag{3.12}$$

Because of (3.12) and Theorem 3.6 of [7], there exists $\varphi\in \boldsymbol{H}(\mathbf{curl},\hat{T})$, such that

$$\hat{\boldsymbol{v}}=\mathbf{curl}\varphi, \tag{3.13}$$

$$\boldsymbol{\varphi}\wedge\hat{\boldsymbol{n}}\big|_{\partial\hat{T}}=0,\quad \mathrm{div}\boldsymbol{\varphi}=0,\quad \int_{\partial\hat{T}}\boldsymbol{\varphi}\cdot\hat{\boldsymbol{n}}\, d\hat{s}=0. \tag{3.14}$$

From the definition of $\boldsymbol{V}(\hat{T})$, there exists $\boldsymbol{\varphi}\in P_2^3(\hat{T})\oplus\hat{b}\tilde{P}_1^{*3}$. Assume $\boldsymbol{\varphi}=\boldsymbol{\omega}+\hat{b}\boldsymbol{\psi}$, $\boldsymbol{\omega}\in P_2^3(\hat{T}), \boldsymbol{\psi}\in\tilde{P}_1^{*3}$, $\boldsymbol{\varphi}\wedge\boldsymbol{n}\big|_{\hat{f}_i}=\boldsymbol{\omega}\wedge\boldsymbol{n}\big|_{\hat{f}_i}, 1\le i\le 4$. Assume $\boldsymbol{\omega}=(w_1,w_2,w_3)$, and for $1\le i\le 3$.

$$w_i=\alpha_{i_0}+\alpha_{i_1}\hat{x}_1+\alpha_{i_2}\hat{x}_2+\alpha_{i_3}\hat{x}_3+\alpha_{i_4}\hat{x}_1^2+\alpha_{i_5}\hat{x}_2^2+\alpha_{i_6}\hat{x}_3^2+\alpha_{i_7}\hat{x}_1\hat{x}_2+\alpha_{i_8}\hat{x}_1\hat{x}_3+\alpha_{i_9}\hat{x}_2\hat{x}_3.$$

On $\hat{f}_1$, $\hat{x}_1=0$, $\hat{\boldsymbol{n}}_1=(1,0,0)$,

$$\boldsymbol{\omega}\wedge\hat{\boldsymbol{n}}\big|_{\hat{f}_1}=\left|\begin{matrix}\vec{i} & \vec{j} & \vec{k}\\ w_1 & w_2 & w_3\\ 1 & 0 & 0\end{matrix}\right|_{\hat{f}_1}=(0,w_3,-w_2)\big|_{\hat{x}_1=0}=0,$$

which leads to

$$\alpha_{i_0}=\alpha_{i_2}=\alpha_{i_3}=\alpha_{i_5}=\alpha_{i_6}=\alpha_{i_9}=0,\qquad i=2,3. \tag{3.15}$$

On $\hat{f}_2$, $\hat{x}_2=0$, $\hat{\boldsymbol{n}}_2=(0,1,0)$,

$$\boldsymbol{\omega}\wedge\hat{\boldsymbol{n}}\big|_{\hat{f}_1}=\left|\begin{matrix}\vec{i} & \vec{j} & \vec{k}\\ w_1 & w_2 & w_3\\ 0 & 1 & 0\end{matrix}\right|_{\hat{f}_2}=(w_3,0,-w_1)\big|_{\hat{x}_2=0}=0,$$

which gives

$$\alpha_{i_0}=\alpha_{i_1}=\alpha_{i_3}=\alpha_{i_4}=\alpha_{i_6}=\alpha_{i_8}=0,\qquad i=1,3. \tag{3.16}$$

On $\hat{f}_3$, $\hat{x}_3=0$, $\hat{\boldsymbol{n}}_3=(0,0,1)$,

$$\boldsymbol{\omega}\wedge\hat{\boldsymbol{n}}\big|_{\hat{f}_3}=\left|\begin{matrix}\vec{i} & \vec{j} & \vec{k}\\ w_1 & w_2 & w_3\\ 0 & 0 & 1\end{matrix}\right|_{\hat{f}_3}=(w_2,-w_1,0)\big|_{\hat{x}_3=0}=0,$$

which leads to

$$\alpha_{i_0}=\alpha_{i_1}=\alpha_{i_2}=\alpha_{i_4}=\alpha_{i_5}=\alpha_{i_7}=0,\qquad i=1,2. \tag{3.17}$$

From (3.15)-(3.17), we obtain

$$w_1=\alpha\hat{x}_2\hat{x}_3,\quad w_2=\beta\hat{x}_1\hat{x}_3,\quad w_3=\gamma\hat{x}_1\hat{x}_2, \tag{3.18}$$

where $\alpha,\beta,\gamma$ are constants. On $\hat{f}_4$, $1-\hat{x}_1-\hat{x}_2-\hat{x}_3=0$, $\hat{\boldsymbol{n}}_4=\frac{1}{\sqrt{3}}(1,1,1)$,

$$\sqrt{3}\boldsymbol{\omega}\wedge\hat{\boldsymbol{n}}\big|_{\hat{f}_4}=\left|\begin{matrix}\vec{i} & \vec{j} & \vec{k}\\ w_1 & w_2 & w_3\\ 0 & 0 & 1\end{matrix}\right|_{\hat{f}_4}=(w_2-w_3,w_3-w_1,w_1-w_2)\big|_{\hat{f}_4}=0,$$

$$w_2-w_3=\beta\hat{x}_1\hat{x}_3-\gamma\hat{x}_1\hat{x}_2=\beta\hat{x}_1(1-\hat{x}_1-\hat{x}_2)-\gamma\hat{x}_1\hat{x}_2\equiv 0.$$

Then $\beta=\gamma=0$,

$$w_3-w_1=\gamma\hat{x}_1\hat{x}_2-\alpha\hat{x}_2\hat{x}_3=\gamma\hat{x}_1\hat{x}_2-\alpha\hat{x}_2(1-\hat{x}_1-\hat{x}_2)\equiv 0,$$

so $\alpha=\gamma=0$. Consequently $\boldsymbol{\omega}=0$. Then $\boldsymbol{\varphi}=\hat{b}\boldsymbol{\psi},\boldsymbol{\psi}\in\tilde{P}_1^{3*}$. Define $\boldsymbol{\psi}=(\psi_1,\psi_2,\psi_3)$,

$$\psi_i=\beta_{i_1}\hat{x}_1+\beta_{i_2}\hat{x}_2+\beta_{i_3}\hat{x}_3,\ 1\le i\le 3. \tag{3.19}$$

$$\begin{aligned}\mathbf{curl}\boldsymbol{\varphi}\big|_{\hat{f}_i}&=\mathbf{curl}(\hat{b}\boldsymbol{\psi})\big|_{\hat{f}_i}\\ &=\Big(\frac{\partial(\hat{b}\psi_3)}{\partial\hat{x}_2}-\frac{\partial(\hat{b}\psi_2)}{\partial\hat{x}_3},\frac{\partial(\hat{b}\psi_1)}{\partial\hat{x}_3}-\frac{\partial(\hat{b}\psi_3)}{\partial\hat{x}_1},\frac{\partial(\hat{b}\psi_2)}{\partial\hat{x}_1}-\frac{\partial(\hat{b}\psi_1)}{\partial\hat{x}_2}\Big)\Big|_{\hat{f}_i}\\ &=\Big(\frac{\partial\hat{b}}{\partial\hat{x}_2}\psi_3-\frac{\partial\hat{b}}{\partial\hat{x}_3}\psi_2,\frac{\partial\hat{b}}{\partial\hat{x}_3}\psi_1-\frac{\partial\hat{b}}{\partial\hat{x}_1}\psi_3,\frac{\partial\hat{b}}{\partial\hat{x}_1}\psi_2-\frac{\partial\hat{b}}{\partial\hat{x}_2}\psi_1\Big)\Big|_{\hat{f}_i},\\ &\triangleq(q_1,q_2,q_3)\triangleq\boldsymbol{Q},\qquad 1\le i\le 4.\end{aligned} \tag{3.20}$$

Since $\hat{b}=\hat{x}_1\hat{x}_2\hat{x}_3(1-\hat{x}_1-\hat{x}_2-\hat{x}_3)$, we have

$$\frac{\partial\hat{b}}{\partial\hat{x}_1}=\hat{x}_2\hat{x}_3(1-\hat{x}_1-\hat{x}_2-\hat{x}_3)-\hat{x}_1\hat{x}_2\hat{x}_3, \tag{3.21a}$$

$$\frac{\partial\hat{b}}{\partial\hat{x}_2}=\hat{x}_1\hat{x}_3(1-\hat{x}_1-\hat{x}_2-\hat{x}_3)-\hat{x}_1\hat{x}_2\hat{x}_3, \tag{3.21b}$$

$$\frac{\partial\hat{b}}{\partial\hat{x}_1}=\hat{x}_1\hat{x}_2(1-\hat{x}_1-\hat{x}_2-\hat{x}_3)-\hat{x}_1\hat{x}_2\hat{x}_3. \tag{3.21c}$$

For (3.9b), we first note

$$\int_{\hat{f}_i} \boldsymbol{Q} \wedge \hat{\boldsymbol{n}}\, d\hat{s} \overset{(3.20)}{=} \int_{\hat{f}_i} \mathbf{curl}\boldsymbol{\varphi} \wedge \hat{\boldsymbol{n}}\, d\hat{s} = \int_{\hat{f}_i} \hat{\boldsymbol{v}} \wedge \hat{\boldsymbol{n}}\, d\hat{s} = 0, \qquad 1 \le i \le 4. \tag{3.22}$$

Its is known that

$$\int_{\hat{f}} \hat{\lambda}_i \hat{\lambda}_j^{\,2} \hat{\lambda}_k = \frac{1!2!1!2!}{6!} |\hat{f}| = \frac{1}{180} |\hat{f}|. \tag{3.23}$$

On $\hat{f}_1$, $\hat{x}_1 = 0$, $\hat{\boldsymbol{n}}_1 = (1,0,0)$,

$$\begin{aligned}
\int_{\hat{f}_1} \boldsymbol{Q} \wedge \hat{\boldsymbol{n}}\, d\hat{s} &= \int_{\hat{f}_1} \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ q_1 & q_2 & q_3 \\ 1 & 0 & 0 \end{matrix} \right|_{\hat{f}_1} d\hat{s} = \int_{\hat{f}_1} (0, q_3, -q_2)\big|_{\hat{x}_1=0}\, d\hat{s} \\
&\overset{(3.20)}{=} \int_{\hat{f}_1} (0, \psi_2 \frac{\partial \hat{b}}{\partial \hat{x}_1}, \psi_3 \frac{\partial \hat{b}}{\partial \hat{x}_1})\big|_{\hat{x}_1=0}\, d\hat{s} \\
&= \int_{\hat{f}_1} (0, \hat{\lambda}_2\hat{\lambda}_3\hat{\lambda}_4(\beta_{2_2}\hat{\lambda}_2 + \beta_{2_3}\hat{\lambda}_3), \hat{\lambda}_2\hat{\lambda}_3\hat{\lambda}_4(\beta_{3_2}\hat{\lambda}_2 + \beta_{3_3}\hat{\lambda}_3))\, d\hat{s} \\
&\overset{(3.23)}{=} -\frac{|\hat{f}_1|}{180}(0, \beta_{2_2} + \beta_{2_3}, \beta_{3_2} + \beta_{3_3}) = 0,
\end{aligned}$$

which gives

$$\beta_{2_2} + \beta_{2_3} = 0, \qquad \beta_{3_2} + \beta_{3_3} = 0. \tag{3.24}$$

On $\hat{f}_2$, $\hat{x}_2 = 0$, $\hat{\boldsymbol{n}}_2 = (0,1,0)$,

$$\begin{aligned}
\int_{\hat{f}_2} \boldsymbol{Q} \wedge \hat{\boldsymbol{n}}\, d\hat{s} &= \int_{\hat{f}_2} \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ q_1 & q_2 & q_3 \\ 0 & 1 & 0 \end{matrix} \right|_{\hat{f}_2} d\hat{s} = \int_{\hat{f}_2} (-q_3, 0, q_1)\big|_{\hat{x}_2=0}\, d\hat{s} \\
&\overset{(3.21)}{=} \int_{\hat{f}_2} (\psi_1 \frac{\partial \hat{b}}{\partial \hat{x}_2}, 0, \psi_3 \frac{\partial \hat{b}}{\partial \hat{x}_2})\big|_{\hat{x}_2=0}\, d\hat{s} \\
&= \int_{\hat{f}_2} (\hat{\lambda}_1\hat{\lambda}_3\hat{\lambda}_4(\beta_{1_1}\hat{\lambda}_1 + \beta_{1_3}\hat{\lambda}_3), 0, \hat{\lambda}_1\hat{\lambda}_3\hat{\lambda}_4(\beta_{3_1}\hat{\lambda}_1 + \beta_{3_3}\hat{\lambda}_4))\, d\hat{s} \\
&= \frac{|\hat{f}_2|}{180}(\beta_{1_2} + \beta_{1_3}, 0, \beta_{3_1} + \beta_{3_3}) = 0,
\end{aligned}$$

which yields

$$\beta_{1_2} + \beta_{1_3} = 0, \qquad \beta_{3_1} + \beta_{3_3} = 0. \tag{3.25}$$

On $\hat{f}_3$, $\hat{x}_3 = 0$, $\hat{\boldsymbol{n}}_1 = (0,0,1)$,

$$\begin{aligned}
\int_{\hat{f}_3} \boldsymbol{Q} \wedge \hat{\boldsymbol{n}}\, d\hat{s} &= \int_{\hat{f}_3} \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ q_1 & q_2 & q_3 \\ 0 & 1 & 0 \end{matrix} \right|_{\hat{f}_3} d\hat{s} = \int_{\hat{f}_3} (q_2, -q_1, 0)\big|_{\hat{x}_3=0}\, d\hat{s} \\
&\overset{(3.21)}{=} \int_{\hat{f}_3} (\psi_1 \frac{\partial \hat{b}}{\partial \hat{x}_3}, \psi_2 \frac{\partial \hat{b}}{\partial \hat{x}_3}, 0)\big|_{\hat{x}_2=0}\, d\hat{s} \\
&= \int_{\hat{f}_3} (\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_4(\beta_{1_1}\hat{\lambda}_1 + \beta_{1_2}\hat{\lambda}_2), \hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_4(\beta_{2_1}\hat{\lambda}_1 + \beta_{2_2}\hat{\lambda}_4), 0)\, d\hat{s} \\
&= \frac{|\hat{f}_2|}{180}(\beta_{1_1} + \beta_{1_2}, \beta_{2_1} + \beta_{2_2}, 0) = 0,
\end{aligned}$$

which yields

$$\beta_{1_1} + \beta_{1_2} = 0, \qquad \beta_{2_1} + \beta_{2_2} = 0. \tag{3.26}$$

On $\hat{f}_4$, $1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3 = 0$, $\hat{\boldsymbol{n}}_4 = \frac{1}{\sqrt{3}}(1,1,1)$,

$$\sqrt{3}\int_{\hat{f}_4} \boldsymbol{Q} \wedge \hat{\boldsymbol{n}}\, d\hat{s} = \int_{\hat{f}_4} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ q_1 & q_2 & q_3 \\ 1 & 1 & 1 \end{vmatrix}_{\hat{f}_4} d\hat{s} = \int_{\hat{f}_4} (q_2 - q_3, q_3 - q_1, q_1 - q_2)\, d\hat{s}$$

$$\overset{(3.21)}{=} \int_{\hat{f}_4} (\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3(-2\psi_1 + \psi_2 + \psi_3), \hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3(\psi_1 - 2\psi_2 + \psi_3), \hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3(\psi_1 + \psi_2 - 2\psi_3))\, d\hat{s}$$
$$= 0,$$

Hence we have

$$-2(\beta_{1_1} + \beta_{1_2} + \beta_{1_3}) + (\beta_{2_1} + \beta_{2_2} + \beta_{2_3}) + (\beta_{3_1} + \beta_{3_2} + \beta_{3_3}) = 0, \tag{3.27}$$

$$-2(\beta_{2_1} + \beta_{2_2} + \beta_{2_3}) + (\beta_{1_1} + \beta_{1_2} + \beta_{1_3}) + (\beta_{3_1} + \beta_{3_2} + \beta_{3_3}) = 0. \tag{3.28}$$

Its follows from the definition of (3.2), that

$$\beta_{1_1} + \beta_{2_2} + \beta_{3_3} = 0. \tag{3.29}$$

From the above (3.24)–(3.29), it yields that

$$A\boldsymbol{\beta} = 0,$$

where $A$ is $9 \times 9$ matrix,

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ -2 & -2 & -2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 & -2 & -2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to show that $\det(A) = 9$. Then

$$\beta_{i_j} = 0, \qquad 1 \le i, j \le 3. \tag{3.30}$$

From (3.19), $\boldsymbol{\psi} = 0$, $\boldsymbol{\varphi} = \hat{b}\boldsymbol{\psi} = 0$, and from (3.13), $\boldsymbol{v} = 0$ is deduced. $\square$

**Remark 3.1.** Firstly, from (3.10), we know that the finite element space of DST20 is conforming in $\boldsymbol{H}(\mathrm{div}, \Omega)$. Secondly, in [30], authors presented a tetrahedral element for Darcy-Stokes problem whose convergent rate is the same as ours, but the dimension of $\boldsymbol{V}(T)$ of their element is 24. So our element is simplified form of [30]. On the other hand, our proof for unique solvability of $\boldsymbol{V}(\hat{T})$ by the degrees of freedom is different from theirs. Thirdly, the above element has the same degrees of freedom as the element in [30], but it has different shape function space, ours is simple and has the explicit expression.

The interpolation operator $\hat{\Pi} : \boldsymbol{H}^1 \cap \boldsymbol{H}_0^1(\widehat{\mathrm{div}}) \rightarrow \boldsymbol{V}(\hat{T})$ is defined via

$$\int_{\hat{f}_i} (\hat{\boldsymbol{v}} - \hat{\Pi}\hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{n}} p \, d\hat{s}, \qquad \forall p \in P_1(\hat{f}_i), \quad 1 \le i \le 4, \tag{3.31a}$$

$$\int_{\hat{f}_i} (\hat{\boldsymbol{v}} - \hat{\Pi}\hat{\boldsymbol{v}}) \wedge \hat{\boldsymbol{n}} \, d\hat{s}, \qquad 1 \le i \le 4. \tag{3.31b}$$

$\Pi_T : \boldsymbol{H}^1 \cap \boldsymbol{H}_0^1(\mathrm{div}) \rightarrow \boldsymbol{V}(T)$ is defined by

$$\int_{f_i} (\boldsymbol{v} - \Pi_T \boldsymbol{v}) \cdot \boldsymbol{n} p \, ds, \qquad \forall p \in P_1(f_i), \quad 1 \le i \le 4, \tag{3.32a}$$

$$\int_{f_i} (\boldsymbol{v} - \Pi_T \boldsymbol{v}) \wedge \boldsymbol{n} \, ds, \qquad 1 \le i \le 4. \tag{3.32b}$$

Define $\Pi_h : \boldsymbol{H}^1 \cap \boldsymbol{H}_0^1(\mathrm{div}) \rightarrow \boldsymbol{V}_h$, $\Pi_h|_T = \Pi_T$, and $\boldsymbol{V}_h|_T = \boldsymbol{V}(T), \forall T \in \mathcal{T}_h$.

Given the shape function space of the pressure of DST20

$$Q(\hat{T}) = P_0(\hat{T}). \tag{3.33}$$

The degrees of freedom are taken as

$$\int_{\hat{T}} \hat{q} \, d\hat{x}. \tag{3.34}$$

The corresponding interpolation operator also being $L^2$-projection on $Q(\hat{T})$ is $P_{\hat{T}} : L^2(\hat{T}) \rightarrow Q(\hat{T})$ satisfying,

$$\int_{\hat{T}} (\hat{q} - P_{\hat{T}}\hat{q}) \, d\hat{s} = 0. \tag{3.35}$$

On the general element $T$, we have

$$||q - P_T q||_{j,T} \lesssim h^{l-j} |q|_{l,T}, \qquad 0 \le j \le l \le 1. \tag{3.36}$$

## 4. Uniformly Convergent of Element DSC20

### 4.1. Uniformly convergence of the discrete Darcy-Stokes system

Now we consider the finite element method for (2.1). Let $\mathcal{T}_h$ be a shape regular triangulation of $\Omega$ with the mesh parameter $h = \max\limits_{T \in \mathcal{T}_h} \{\text{diameter of } T\}$, $\Omega = \bigcup_{T \in \mathcal{T}_h} T$, $T$ be an element. The discrete problem of (2.1) is

$$\begin{cases} a_{\varepsilon,h}(\boldsymbol{u}_h, \boldsymbol{v}_h) + (p_h, \mathrm{div}\boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ (\mathrm{div}\boldsymbol{u}_h, q_h) = 0, & \forall q_h \in Q_h, \end{cases} \tag{4.1}$$

where

$$a_{\varepsilon,h}(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{u}_h, \boldsymbol{v}_h) + \varepsilon^2 \sum_{\partial T} \int_{\partial T} \boldsymbol{D}\boldsymbol{u}_h : \boldsymbol{D}\boldsymbol{v}_h \, dx.$$

The discrete energy norm is defined by

$$|||\boldsymbol{v}_h|||_{\varepsilon,h}^2 = ||\boldsymbol{v}_h||_{0,\Omega}^2 + ||\mathrm{div}\boldsymbol{v}_h||_{0,\Omega}^2 + \varepsilon^2 \sum_{T \in \mathcal{T}} |\boldsymbol{v}_h|_{1,T}^2. \tag{4.2}$$

**Lemma 4.1.** *It holds that* $\mathrm{div}\, \boldsymbol{V}_h = Q_h$.

*Proof.* For any $\boldsymbol{v}_h \in \boldsymbol{V}_h$, because $\mathrm{div}\boldsymbol{v}_h\big|_T = \mathrm{div}\,\boldsymbol{V}(T) = P_0(T)$, and

$$\int_\Omega \mathrm{div}\boldsymbol{v}_h \; dx = \int_{\partial\Omega} \boldsymbol{v}_h \cdot \boldsymbol{n} \; dx = 0,$$

it implies that $\mathrm{div}\,\boldsymbol{V}_h \subset Q_h$. On the other hand, for all $q_h \in Q_h \subset \boldsymbol{L}_0^2(\Omega)$, there exists $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, $\mathrm{div}\boldsymbol{v} = q$, we obtain $\Pi_h \boldsymbol{v} \in V_h$,

$$\int_\Omega \mathrm{div}\Pi_h \boldsymbol{v} dx = \int_{\partial\Omega} \Pi_h \boldsymbol{v} \cdot \boldsymbol{n} ds = \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n} \, ds = \int_\Omega \mathrm{div}\boldsymbol{v} \, dx,$$

which implies that

$$\mathrm{div}\Pi_h \boldsymbol{v} = P_h \mathrm{div}\boldsymbol{v} = P_h q_h \overset{q_h \in Q_h}{=} q_h, \; q_h \in \mathrm{div}\,\boldsymbol{V}_h, \tag{4.3}$$

that is $Q_h \subset \mathrm{div}\,\boldsymbol{V}_h$. □

We shall discuss the proper uniform inf-sup condition of the discrete Darcy-Stokes system (4.1). For all $q_h \in Q_h \subset \boldsymbol{L}_0^2(T)$, there exists $\boldsymbol{v} \in \boldsymbol{H}_0^1(T), \mathrm{div}\boldsymbol{v} = q_h, ||\boldsymbol{v}||_1 \leq c||q_h||_0$. Consequently

$$\Pi_h \boldsymbol{v} \in \boldsymbol{V}_h, \; \mathrm{div}\Pi_h \boldsymbol{v} = q_h, \; ||\Pi_h \boldsymbol{v}||_{1,h} \leq ||\boldsymbol{v}||_1 + ||\boldsymbol{v} - \Pi_h \boldsymbol{v}||_{1,h} \leq c||\boldsymbol{v}||_1.$$

Note that

$$|||\Pi_h \boldsymbol{v}|||_{\varepsilon,h}^2 = ||\Pi_h \boldsymbol{v}||_0^2 + ||\mathrm{div}\Pi_h \boldsymbol{v}||_0^2 + \varepsilon^2 |\Pi_h \boldsymbol{v}|_{1,h}^2 \leq c||\Pi_h \boldsymbol{v}||_{1,h}^2 \leq c||\boldsymbol{v}||_1^2 \leq c||q_h||_0^2,$$

we get

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{(q_h, \mathrm{div}\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_{\varepsilon,h}} \geq \frac{(q_h, \mathrm{div}\Pi_h \boldsymbol{v})}{|||\Pi_h \boldsymbol{v}|||_{\varepsilon,h}} \geq \frac{||q_h||_0^2}{c||q_h||_0} = \frac{1}{c}||q_h||_0,$$

and then the assertion is proved.

Since $\mathrm{div}\,\boldsymbol{V}_h = Q_h$, we obtain

$$\boldsymbol{Z}_h = \Big\{\boldsymbol{v}_h \in \boldsymbol{V}_h, \; (q_h, \mathrm{div}\boldsymbol{v}_h) = 0, \; \forall q_h \in Q_h\Big\} = \Big\{\boldsymbol{v}_h \in \boldsymbol{V}_h; \; \mathrm{div}\boldsymbol{v}_h = 0\Big\}.$$

Then for all $\boldsymbol{v}_h \in \boldsymbol{Z}_h$, we have

$$\begin{aligned} a_{\varepsilon,h}(\boldsymbol{v}_h, \boldsymbol{v}_h) &= ||\boldsymbol{v}_h||_0^2 + \varepsilon^2 |\boldsymbol{v}_h|_{1,h}^2 \\ &= ||\boldsymbol{v}_h||_0^2 + \varepsilon^2 |\boldsymbol{v}|_{1,h}^2 + ||\mathrm{div}\boldsymbol{v}_h||_0^2 \equiv |||\boldsymbol{v}_h|||_{\varepsilon,h}^2. \end{aligned} \tag{4.4}$$

Combining our previous relations, the discrete system (4.1) has one and only one solution.

### 4.2. Uniform error estimates for the Darcy-Stokes system

**Lemma 4.2.** ([11]) *The discrete problem* (4.1) *has following error estimates*

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\varepsilon,h} \leq 2|||\boldsymbol{u} - \Pi_h \boldsymbol{u}|||_{\varepsilon,h} + \sup_{\boldsymbol{w}_h \in \boldsymbol{V}_h} \frac{|E_{\varepsilon,h}(\boldsymbol{u}, \boldsymbol{w}_h)|}{|||\boldsymbol{w}_h|||_{\varepsilon,h}}, \tag{4.5}$$

$$||p - p_h||_{0,\Omega} \leq c\Bigg(|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\varepsilon,h} + ||p - P_h p||_{0,\Omega} + \sup_{\boldsymbol{w}_h \in \boldsymbol{V}_h} \frac{|E_{\varepsilon,h}(\boldsymbol{u}, \boldsymbol{w}_h)|}{|||\boldsymbol{w}_h|||_{\varepsilon,h}}\Bigg), \tag{4.6}$$

*where $c$ is independent of $\varepsilon, h, \boldsymbol{f}$ and*

$$E_{\varepsilon,h}(\boldsymbol{u}, \boldsymbol{w}_h) = \sum_{T \in T_h} \varepsilon^2 \int_{\partial T} \frac{\partial \boldsymbol{u}}{\partial n} \cdot \boldsymbol{w}_h \, dx. \tag{4.7}$$

In the following, we estimate $|||\boldsymbol{u}-\Pi_h\boldsymbol{u}|||_{\varepsilon,h}$ and $\sup\limits_{\boldsymbol{w}_h\in\boldsymbol{V}_h}|E_{\varepsilon,h}(\boldsymbol{u},\boldsymbol{w}_h)|/|||\boldsymbol{w}_h|||_{\varepsilon,h}$. By the standard interpolation theory of [5], we have

$$|\boldsymbol{v}-\Pi_T\boldsymbol{v}|_{j,T}\le ch^{l-j}|\boldsymbol{v}|_{l,T},\qquad 1\le l\le 2,\ 0\le j\le 1, \tag{4.8}$$

$$||q-I_hq||_{0,T}\le ch|q|_{1,T}. \tag{4.9}$$

Firstly, we estimate $||\boldsymbol{u}-\Pi_h\boldsymbol{u}||_{0,T}$. Define

$$P_i\boldsymbol{v}=\frac{1}{|f_i|}\int_{f_i}\boldsymbol{v}\,ds,\qquad \boldsymbol{v}=\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}},\qquad P_T\boldsymbol{v}=\frac{1}{|T|}\int_T\boldsymbol{v}\,ds.$$

It can be verified that

$$\begin{aligned}&||\boldsymbol{u}-\Pi_h\boldsymbol{u}||_{0,T}\le ch||\hat{\boldsymbol{u}}-\hat{\Pi}\hat{\boldsymbol{u}}||_{0,\hat{T}} \qquad (4.10)\\ &\le ch(||(\hat{I}-\hat{\Pi})(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)||_{0,\hat{T}}+||\hat{\boldsymbol{u}}^0-\hat{\Pi}\hat{\boldsymbol{u}}^0||_{0,\hat{T}})\\ &=ch(||(\hat{I}-\hat{\Pi})((\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)-P_{\hat{T}}(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0))||_{0,\hat{T}}+||\hat{\boldsymbol{u}}^0-\hat{\Pi}\hat{\boldsymbol{u}}^0||_{0,\hat{T}})\\ &\le ch(||(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)-P_{\hat{T}}(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)||_{0,\hat{T}}+||\hat{\Pi}((\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)-P_{\hat{T}}(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0))||_{0,\hat{T}}+||\hat{\boldsymbol{u}}^0-\hat{\Pi}\hat{\boldsymbol{u}}^0||_{0,\hat{T}}).\end{aligned}$$

Note that

$$\begin{aligned}&||(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)-P_{\hat{T}}(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)||_{0,\hat{T}}\\ &\quad=||(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)-P_{\hat{T}}(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)||^{\frac12}_{0,\hat{T}}||(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)-P_{\hat{T}}(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)||^{\frac12}_{0,\hat{T}}\\ &\quad\le||\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0||^{\frac12}_{0,\hat{T}}|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0|^{\frac12}_{1,\hat{T}}\\ &\quad\le ch^{-\frac12}||\boldsymbol{u}-\boldsymbol{u}^0||^{\frac12}_{0,T}|\boldsymbol{u}-\boldsymbol{u}^0|^{\frac12}_{1,T}\\ &\quad\overset{(2.4)(2.5)}{\le} ch^{-\frac12}\cdot\varepsilon^{\frac14}||\boldsymbol{f}||^{\frac12}_{1,T}\cdot\varepsilon^{-\frac14}||\boldsymbol{f}||^{\frac12}_{1,T}\le ch^{-\frac12}||\boldsymbol{f}||_{1,T}, \qquad (4.11)\\ &||\hat{\Pi}((\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)-P_{\hat{T}}(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0))||_{0,\hat{T}}\\ &\quad\overset{(4.8)}{\le} c||(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)-P_{\hat{T}}(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)||^{\frac12}_{0,\hat{T}}||(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)-P_{\hat{T}}(\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}^0)||^{\frac12}_{0,\hat{T}}\\ &\quad\le ch^{-\frac12}||\boldsymbol{f}||_{1,T}. \qquad (4.12)\end{aligned}$$

Then we have

$$||\hat{\boldsymbol{u}}^0-\hat{\Pi}\hat{\boldsymbol{u}}^0||_{0,\hat{T}}\overset{(4.8)}{\le}c|\hat{\boldsymbol{u}}^0|_{1,\hat{T}}\le c|\boldsymbol{u}^0|_{1,T}\overset{(2.4)}{\le}c||\boldsymbol{f}||_1. \tag{4.13}$$

Substituting (4.11)–(4.13) into (4.10) gives

$$||\boldsymbol{u}-\Pi_h\boldsymbol{u}||_{0,T}\le ch^{\frac12}||\boldsymbol{f}||_{1,T}. \tag{4.14}$$

Secondly, we note

$$||\mathrm{div}(\boldsymbol{u}-\Pi_h\boldsymbol{u})||_0=0. \tag{4.15}$$

Thirdly, by noting that

$$\begin{aligned}&\varepsilon||\boldsymbol{D}(\boldsymbol{u}-\Pi_h\boldsymbol{u})||_{0,T}\\ &=\varepsilon|\boldsymbol{u}-\Pi_h\boldsymbol{u}|_{1,T}\le c\varepsilon h^{\frac12}|\boldsymbol{u}|^{\frac12}_{1,T}|\boldsymbol{u}|^{\frac12}_{2,T}\overset{(2.5)}{\le}ch^{\frac12}||\boldsymbol{f}||_{1,T},\end{aligned}$$

we obtain

$$\varepsilon||\boldsymbol{D}(\boldsymbol{u}-\Pi_h\boldsymbol{u})||_{0,h}\le ch^{\frac{1}{2}}||\boldsymbol{f}||_1. \tag{4.16}$$

Using (4.14)–(4.16) leads to

$$\begin{aligned}&|||\boldsymbol{u}-\Pi_h\boldsymbol{u}|||_{\varepsilon,h}\\ =&\Big(||\boldsymbol{u}-\Pi_h\boldsymbol{u}||_0^2+||\mathrm{div}\boldsymbol{u}-\mathrm{div}\Pi_h\boldsymbol{u}||_0^2+\varepsilon^2||\boldsymbol{D}(\boldsymbol{u}-\Pi_h\boldsymbol{u})||_{0,h}^2\Big)^{\frac{1}{2}}\le ch^{\frac{1}{2}}||\boldsymbol{f}||_1.\end{aligned} \tag{4.17}$$

Lastly, we estimate $\sup\limits_{\boldsymbol{w}_h\in\boldsymbol{V}_h}|E_{\varepsilon,h}(\boldsymbol{u},\boldsymbol{w}_h)|/|||\boldsymbol{w}_h|||_{\varepsilon,h}$. It can be verified that

$$\begin{aligned}&|E_{\varepsilon,h}(\boldsymbol{u},\boldsymbol{w}_h)|=\Big|\sum_{T\in\mathcal{T}_h}\varepsilon^2\int_{\partial T}\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\cdot\boldsymbol{w}_h\,ds\Big|=\Big|\sum_{T\in\mathcal{T}_h}\varepsilon^2\sum_{i=1}^{6}\int_{\partial f_i}\boldsymbol{v}\cdot\boldsymbol{w}_h\,ds\Big|\\
&=\varepsilon^2\Big|\sum_{T\in\mathcal{T}_h}\sum_{i=1}^{6}\int_{\partial f_i}(\boldsymbol{v}-P_T\boldsymbol{v})\cdot(\boldsymbol{w}_h-P_i\boldsymbol{w}_h)\Big|\\
&\le c\varepsilon^2\sum_{T\in\mathcal{T}_h}\sum_{i=1}^{6}|f_i|\cdot||\hat{\boldsymbol{v}}-P_{\hat{T}}\hat{\boldsymbol{v}}||_{0,\hat{f}_i}||\hat{\boldsymbol{w}}_h-\hat{P}_i\hat{\boldsymbol{w}}_h||_{0,\hat{f}_i}\\
&\le c\varepsilon^2\sum_{T\in\mathcal{T}_h}h^2||\hat{\boldsymbol{v}}-P_{\hat{T}}\hat{\boldsymbol{v}}||_{0,\hat{T}}^{\frac{1}{2}}||\hat{\boldsymbol{v}}-P_{\hat{I}}\hat{\boldsymbol{v}}||_{1,\hat{T}}^{\frac{1}{2}}||\hat{\boldsymbol{w}}_h-\hat{P}_i\hat{\boldsymbol{w}}_h||_{1,\hat{T}}\le c\varepsilon^2\sum_{T\in\mathcal{T}_h}h^2||\hat{\boldsymbol{v}}||_{0,\hat{T}}^{\frac{1}{2}}|\hat{\boldsymbol{v}}|_{1,\hat{T}}^{\frac{1}{2}}|\hat{\boldsymbol{w}}_h|_{1,\hat{T}}\\
&\le c\varepsilon^2\sum_{T\in\mathcal{T}_h}h^2\cdot h^{-\frac{3}{4}}||\boldsymbol{v}||_{0,T}^{\frac{1}{2}}\cdot h^{-\frac{3}{4}}h^{\frac{1}{2}}|\boldsymbol{v}|_{1,T}^{\frac{1}{2}}\cdot h^{-\frac{3}{2}}h|\boldsymbol{w}_h|_{1,T}\le c\varepsilon^2h^{\frac{1}{2}}\sum_{T\in\mathcal{T}_h}|\boldsymbol{u}|_{1,T}^{\frac{1}{2}}|\boldsymbol{u}|_{2,T}^{\frac{1}{2}}|\boldsymbol{v}_h|_{1,T}\\
&\le c\varepsilon^2h^{\frac{1}{2}}|\boldsymbol{u}|_1^{\frac{1}{2}}|\boldsymbol{u}|_2^{\frac{1}{2}}|\boldsymbol{w}_h|_{1,T}\le ch^{\frac{1}{2}}\Big(\varepsilon^{-\frac{1}{4}}||\boldsymbol{f}||_1^{\frac{1}{2}}\cdot\varepsilon^{-\frac{3}{4}}||\boldsymbol{f}||_1^{\frac{1}{2}}\Big)\varepsilon^2|\boldsymbol{w}_h|_{1,h}\le ch^{\frac{1}{2}}||\boldsymbol{f}||_1||\boldsymbol{w}_h||_{\varepsilon,h},\end{aligned} \tag{4.18}$$

which yields

$$\sup_{\boldsymbol{w}_h\in\boldsymbol{V}_h}\frac{|E_{\varepsilon,h}(\boldsymbol{u},\boldsymbol{w}_h)|}{|||\boldsymbol{w}_h|||_{\varepsilon,h}}\le ch^{\frac{1}{2}}||\boldsymbol{f}||_1. \tag{4.19}$$

**Theorem 4.1.** *Suppose $\boldsymbol{u}$ and $\boldsymbol{u}_h$ are the solution of (2.1) and (4.1), $\mathcal{T}_h$ is a regular division of $\Omega$ into rectangle elements, then the discrete problem has a unique solution and*

$$|||\boldsymbol{u}-\boldsymbol{u}_h|||_{\varepsilon,h}+||p-p_h||_0\le ch^{\frac{1}{2}}||\boldsymbol{f}||_1, \tag{4.20}$$

*where $c$ is independent of $\varepsilon,h$ and $\boldsymbol{u}$.*

*Proof.* Combining (4.5), (4.17) and (4.19), we obtain

$$|||\boldsymbol{u}-\boldsymbol{u}_h|||_{\varepsilon,h}\le ch^{\frac{1}{2}}||\boldsymbol{f}||_1, \tag{4.21}$$

$$\begin{aligned}&(p_h-I_hp,\mathrm{div}\boldsymbol{v}_h)=(p_h-p,\mathrm{div}\boldsymbol{v}_h)+(p-I_hp,\mathrm{div}\boldsymbol{v}_h)\\
&\quad=a_{\varepsilon,h}(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h)-E_{\varepsilon,h}(\boldsymbol{u},\boldsymbol{v}_h)+(p-I_hp,\mathrm{div}\boldsymbol{v}_h)\\
&\quad\le|||\boldsymbol{u}-\boldsymbol{u}_h|||_{\varepsilon,h}|||\boldsymbol{v}_h|||_{\varepsilon,h}+|E_{\varepsilon,h}(\boldsymbol{u},\boldsymbol{v}_h)|+||p-I_hp||_0|||\boldsymbol{v}_h|||_{\varepsilon,h}.\end{aligned} \tag{4.22}$$

Consequently, we obtain

$$\begin{aligned}\beta||p_h-I_hp||_0&\le\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{(p_h-I_hp,div\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_{\varepsilon,h}}\\
&\overset{(4.22)}{\le}|||\boldsymbol{u}-\boldsymbol{u}_h|||_{\varepsilon,h}+\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{|E_{\varepsilon,h}(\boldsymbol{u},\boldsymbol{v}_h)|}{|||\boldsymbol{v}_h|||_{\varepsilon,h}}+||p-I_hp||_0\\
&\overset{(4.19)(4.21)}{\le}ch^{\frac{1}{2}}||\boldsymbol{f}||_1+ch|p|_1\overset{(4.4)(4.7)}{\le}ch^{\frac{1}{2}}||\boldsymbol{f}||_1,\end{aligned} \tag{4.23}$$

which leads to (4.20). □

## 5. The Discrete de Rham Complex Corresponding to DST20

The purpose of this section is to present de Rham complexes in three space dimensions corresponding to DST20 elements. In three space dimensions, the Sobolev space version of the de Rham complex can be written in the form

$$R \xrightarrow{\subset} H^2 \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl}) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0. \tag{5.1}$$

A corresponding discrete de Rham complex is the form

$$R \xrightarrow{\subset} S_h \xrightarrow{\mathbf{grad}} \boldsymbol{W}_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\mathrm{div}} Q_h \longrightarrow 0, \tag{5.2}$$

where $S_h$, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$ and $Q_h$ are conforming or nonconforming finite element spaces of $H^2(\Omega)$, $\boldsymbol{H}(\mathbf{curl},\Omega)$, $\boldsymbol{H}(\mathrm{div},\Omega)$ and $L^2(\Omega)$, respectively.

Note that (5.2) is an exact de Rham complex means that:

(1) The composition of two consecutive maps is zero, that is

$$\mathbf{curl}\mathbf{grad} = 0, \qquad \mathrm{div}\mathbf{curl} = 0. \tag{5.3}$$

Obviously, (5.3) holds.

(2) If the domain $\Omega$ is simply connected, the range of each map is exactly the null space of the succeeding map, that is

$$\mathrm{Range}(\mathbf{grad}) = \mathrm{Ker}(\mathbf{curl}),\ \mathrm{Range}(\mathbf{curl}) = \mathrm{Ker}(\mathrm{div}), \tag{5.4}$$

where

$$\begin{aligned}
&\mathrm{Range}(\mathbf{grad}) = \Big\{\boldsymbol{w}_h \in \boldsymbol{W}_h;\ \exists s_h \in S_h \text{ such that } \boldsymbol{w}_h = \mathbf{grad}s_h\Big\},\\
&\mathrm{Ker}(\mathbf{curl}) = \Big\{\boldsymbol{w}_h \in \boldsymbol{W}_h;\ \mathbf{curl}\boldsymbol{w}_h = 0\Big\},\\
&\mathrm{Range}(\mathbf{curl}) = \Big\{\boldsymbol{v}_h \in \boldsymbol{V}_h;\ \exists \boldsymbol{w}_h \in \boldsymbol{W}_h \text{ such that } \boldsymbol{v}_h = \mathbf{curl}\boldsymbol{w}_h\Big\},\\
&\mathrm{Ker}(\mathrm{div}) = \Big\{\boldsymbol{v}_h \in \boldsymbol{V}_h;\ \mathrm{div}\boldsymbol{v}_h = 0\Big\}.
\end{aligned}$$

(3) Let $\Pi_h^S : H^2(\Omega) \to S_h$, $\Pi_h^W : \boldsymbol{H}^1(\Omega) \to \boldsymbol{W}_h$, $\Pi_h^V : \boldsymbol{H}^1(\Omega) \to \boldsymbol{V}_h$, $\Pi_h^Q : L^2(\Omega) \to Q_h$ be the interpolation operators determined by the finite element space $S_h$, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$ and $Q_h$, respectively. Then the following diagram commutes:

$$\begin{array}{ccccccccccc}
R & \xrightarrow{\subset} & H^2 & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & 0\\
 & & \Pi_h^S\big\downarrow & & \Pi_h^W\big\downarrow & & \Pi_h^V\big\downarrow & & \Pi_h^Q\big\downarrow & & \\
R & \longrightarrow & S_h & \xrightarrow{\mathbf{grad}} & \mathbf{W}_h & \xrightarrow{\mathbf{curl}} & \mathbf{V}_h & \xrightarrow{\mathrm{div}} & Q_h & \longrightarrow & 0
\end{array}$$

In other words, the following identities hold:

$$\mathbf{grad}\Pi_h^S = \Pi_h^W\mathbf{grad}, \quad \mathbf{curl}\Pi_h^W = \Pi_h^V\mathbf{curl}, \quad \mathrm{div}\Pi_h^V = \Pi_h^Q\mathrm{div}. \tag{5.5}$$

First we construct the finite element spaces $S(T)$ and $\boldsymbol{W}(T)$. We define

$$S(T) = P_2(T) + bP_1(T). \tag{5.6}$$

The degrees of freedom are

$$s(a_i), \qquad 1 \le i \le 4, \tag{5.7a}$$

$$\int_{l_i} s(x)\, dl, \qquad 1 \le i \le 6, \tag{5.7b}$$

$$\int_{f_i} \frac{\partial s}{\partial n}\, ds, \qquad 1 \le i \le 4. \tag{5.7c}$$

This element was presented in [30], also see [9]. As $S(T)$ is defined by (5.6), this element is also $H^1$-conforming and $H^2$-average conforming. We define

$$\boldsymbol{W}(T) = \boldsymbol{N}_1(T) \oplus \mathbf{grad}(bP_1(T)) \oplus b\tilde{P}_1^{3*}(T), \tag{5.8}$$

where

$$\boldsymbol{N}_1(T) = P_1^3(T) \oplus \boldsymbol{S}_2(T), \qquad \boldsymbol{S}_2(T) = \Big\{ \boldsymbol{w} \in \tilde{P}_2^3(T); \boldsymbol{w} \cdot \boldsymbol{r} = 0, \boldsymbol{r} = (x_1, x_2, x_3)^T \Big\},$$

$N_1(T)$ is presented in [21], dim$\mathbf{N}_1(T)$=20, and

$$\tilde{P}_1^{*3}(T) = \Big\{ \boldsymbol{w} \in \tilde{P}_1^3(T);\ \operatorname{div}\boldsymbol{w} = 0 \Big\}.$$

We easily get $\dim \boldsymbol{W}(T) = 20 + 4 + 8 = 32$.

The degrees of freedom are

$$\int_{l_i} \boldsymbol{w} \cdot \boldsymbol{t} p\, dl, \qquad \forall p \in P_1(l_i),\ 1 \le i \le 6, \tag{5.9a}$$

$$\int_{f_i} \boldsymbol{w} \wedge \boldsymbol{n} ds, \qquad 1 \le i \le 4, \tag{5.9b}$$

$$\int_{f_i} \boldsymbol{w} \cdot \boldsymbol{n} ds, \qquad 1 \le i \le 4, \tag{5.9c}$$

$$\int_{f_i} \mathbf{curl}\boldsymbol{w} \wedge \boldsymbol{n} ds, \qquad 1 \le i \le 4. \tag{5.9d}$$

**Theorem 5.1.** *The element of $\boldsymbol{W}(T)$ are uniquely determined by the 32 degrees of freedom given by (5.9).*

*Proof.* Suppose that $\boldsymbol{w}(x) \in \boldsymbol{W}(T)$ and the degrees of freedom of $\boldsymbol{w}(x)$ are zero, i.e.,

$$\int_{l_i} \boldsymbol{w} \cdot \boldsymbol{t} p\, dl = 0, \qquad \forall p \in P_1(l_i),\ 1 \le i \le 6, \tag{5.10a}$$

$$\int_{f_i} \boldsymbol{w} \wedge \boldsymbol{n} ds = 0, \qquad 1 \le i \le 4, \tag{5.10b}$$

$$\int_{f_i} \boldsymbol{w} \cdot \boldsymbol{n} ds = 0, \qquad 1 \le i \le 4, \tag{5.10c}$$

$$\int_{f_i} \mathbf{curl}\boldsymbol{w} \wedge \boldsymbol{n} ds = 0, \qquad 1 \le i \le 4. \tag{5.10d}$$

Let

$$\boldsymbol{w}(x)=\boldsymbol{w}_1(x)\oplus\mathbf{grad}(bp)+b\boldsymbol{q},$$

where $\boldsymbol{w}_1(x)\in\boldsymbol{N}_1(T),\ p\in P_1(T),\ \boldsymbol{q}\in\tilde{P}_1^{3*}(T)$. Obviously,

$$\mathbf{grad}(bp)\cdot\boldsymbol{t}|_{l_i}=0,\qquad b\boldsymbol{q}\cdot\boldsymbol{t}|_{l_i}=0,\qquad 1\le i\le 6. \tag{5.11a}$$

$$b\boldsymbol{q}\wedge\boldsymbol{n}|_{f_i}=0,\qquad 1\le i\le 4. \tag{5.11b}$$

We have

$$\mathbf{grad}(bp)\wedge\boldsymbol{n}|_{f_i}=0,\ 1\le i\le 4. \tag{5.12}$$

Then by (5.12), (5.11a) and (5.11b), we have

$$\begin{aligned}&\int_{l_i}\boldsymbol{w}\cdot\boldsymbol{t}p\,dl=0\quad\Longleftrightarrow\quad\int_{l_i}\boldsymbol{w}_1\cdot\boldsymbol{t}p\,dl=0,\qquad\forall p\in P_1(l_i),\ 1\le i\le 6,\\&\int_{f_i}\boldsymbol{w}\wedge\boldsymbol{n}\,ds=0\quad\Longleftrightarrow\quad\int_{f_i}\boldsymbol{w}_1\wedge\boldsymbol{n}\,ds=0,\qquad 1\le i\le 4.\end{aligned}$$

By Theorem 1 of [25], we have

$$\boldsymbol{w}_1(x)\wedge\boldsymbol{n}|_{f_i}=0,\qquad 1\le i\le 4. \tag{5.13}$$

$$\boldsymbol{w}_1(x)=0. \tag{5.14}$$

Since

$$\begin{aligned}&b\boldsymbol{q}\cdot\boldsymbol{n}|_{f_i}=0,\qquad 1\le i\le 4,\\&\int_{f_i}\boldsymbol{w}\cdot\boldsymbol{n}\,ds=0\quad\Longleftrightarrow\quad\int_{f_i}\mathbf{grad}(bp)\cdot\boldsymbol{n}\,ds=0,\qquad 1\le i\le 4,\\&\int_{f_i}\mathbf{grad}(bp)\cdot\boldsymbol{n}\,ds=\int_{f_i}p\frac{\partial b}{\partial n}\,ds=0,\end{aligned}$$

it is easy to check that $\dfrac{\partial b}{\partial n}\Big|_{f_i}>0$ in $\mathring{f}$, $1\le i\le 4$. Hence there exists $b_i\in\mathring{f}_i$ such that $p(b_i)=0$, $1\le i\le 4$. Consequently,

$$p(x)=0. \tag{5.15}$$

Now

$$\int_{f_i}\mathbf{curl}\boldsymbol{w}\wedge\boldsymbol{n}\,ds=0\quad\Longleftrightarrow\quad\int_{f_i}\mathbf{curl}(b\boldsymbol{q})\wedge\boldsymbol{n}\,ds=0.$$

By the proof of Lemma 3.2, it holds

$$\boldsymbol{q}(x)=0. \tag{5.16}$$

From (5.14)–(5.16), we get $\boldsymbol{w}(x)=0$. □

**Remark 5.1.** From (5.11), (5.13) and (5.9), we know that

$$\boldsymbol{w}(x)\wedge\boldsymbol{n}\Big|_{f_i}=0,\qquad\int_{f_i}\boldsymbol{w}\cdot\boldsymbol{n}\,ds=0,\qquad 1\le i\le 3.$$

Hence $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming and $\boldsymbol{H}^1$-average conforming.

**Theorem 5.2.** *The finite element spaces $S(T)$, $\boldsymbol{W}(T)$, $\boldsymbol{V}(T)$ and $Q(T)$ defined by* (5.6), (5.8), (3.1) *and* (3.33)*, respectively, form an exact discrete de Rham complex, i.e.,* (5.4) *and* (5.5) *hold.*

*Proof.* (1) We prove (5.4). Obviously, Range(**grad**) $\subset$ Ker(**curl**). Conversely, suppose $\boldsymbol{w}(x) \in$ Ker(**curl**), i.e.

$$\boldsymbol{w}(x) \in \boldsymbol{W}(T), \ \mathbf{curl}\boldsymbol{w} = 0.$$

Let

$$\boldsymbol{w}(x) = \boldsymbol{w}_1(x) + \boldsymbol{s}(x) + \mathbf{grad}(bp) + bq,$$

where $\boldsymbol{w}_1(x) \in \tilde{P}_1^3(T)$, $\boldsymbol{s}(x) \in \boldsymbol{S}_2(T)$, $p \in P_1(T)$, $\boldsymbol{q} \in \tilde{P}_1^{3*}(T)$. Then

$$\mathbf{curl}\boldsymbol{w}(x) = \mathbf{curl}\boldsymbol{w}_1(x) + \mathbf{curl}\boldsymbol{s}(x) + \mathbf{curl}(b\boldsymbol{q}) = 0.$$

Comparing the coefficients of terms with degrees higher than 3, we get

$$\boldsymbol{q}(x) = 0.$$

From [9], $\boldsymbol{S}_2(T) = span\{\boldsymbol{q}_1, \cdots, \boldsymbol{q}_8\}$, where

$$\boldsymbol{q}_1 = \begin{pmatrix} 0 \\ -x_2x_3 \\ x_2^2 \end{pmatrix}, \quad \boldsymbol{q}_2 = \begin{pmatrix} 0 \\ -x_3^2 \\ x_2x_3 \end{pmatrix}, \quad \boldsymbol{q}_3 = \begin{pmatrix} -x_1x_3 \\ 0 \\ x_1^2 \end{pmatrix}, \quad \boldsymbol{q}_4 = \begin{pmatrix} x_2^2 \\ 0 \\ -x_1x_3 \end{pmatrix},$$
$$\boldsymbol{q}_5 = \begin{pmatrix} -x_1x_2 \\ x_1^2 \\ 0 \end{pmatrix}, \quad \boldsymbol{q}_6 = \begin{pmatrix} -x_2^2 \\ x_1x_2 \\ 0 \end{pmatrix}, \quad \boldsymbol{q}_7 = \begin{pmatrix} x_2x_3 \\ -x_1x_3 \\ 0 \end{pmatrix}, \quad \boldsymbol{q}_8 = \begin{pmatrix} 0 \\ x_1x_3 \\ -x_1x_2 \end{pmatrix}.$$

Let $\boldsymbol{s}(x) = \sum_{i=1}^{8} \alpha_i \boldsymbol{q}_i$. We have

$$\mathbf{curl}\boldsymbol{s}(x) = \begin{pmatrix} (\alpha_7 - 2\alpha_8)x_1 + 3\alpha_1x_2 + 3\alpha_2x_3 \\ 3\alpha_3x_1 + (\alpha_7 + \alpha_8)x_2 + 3\alpha_4x_3 \\ 3\alpha_5x_1 + 3\alpha_6x_2 + (-2\alpha_7 + \alpha_8)x_3 \end{pmatrix}.$$

From $\mathbf{curl}\boldsymbol{w}_1(x) + \mathbf{curl}\boldsymbol{s}(x) = 0$, we have $\boldsymbol{s}(x) = 0$. Then

$$\mathbf{curl}\boldsymbol{w}(x) = 0 \quad \Longleftrightarrow \quad \mathbf{curl}\boldsymbol{w}_1(x) = 0.$$

Let $\boldsymbol{w}_1(x) = \boldsymbol{w}_{10}(x) + \boldsymbol{w}_{11}(x)$, where

$$\begin{aligned} &\boldsymbol{w}_{10}(x) = (\beta_{10}, \beta_{20}, \beta_{30})^\top, \\ &\boldsymbol{w}_{11}(x) = \left( \sum_{j=1}^{3} \beta_{1j}x_3, \sum_{j=1}^{3} \beta_{2j}x_j, \sum_{j=1}^{3} \beta_{3j}x_3 \right)^\top, \\ &\mathbf{curl}\boldsymbol{w}_1(x) = (\beta_{32} - \beta_{23}, \beta_{13} - \beta_{31}, \beta_{12} - \beta_{21})^\top = 0. \end{aligned}$$

We then have

$$\beta_{i,i+1} = \beta_{i+1,i}, \quad 1 \le i \le 3.$$

Take $s(x) = s_1(x) + bp(x)$ with

$$s_1(T) = a_0 + \sum_{j=1}^{3} a_jx_j + \sum_{j=i}^{3} b_jx_j^2 + \sum_{j=1}^{3} c_jx_jx_{j+1},$$

where $a_i = \beta_{i0}$, $b_i = \beta_{ii}/2$, $c_i = \beta_{i,i+1}$, $1 \le i \le 3$. We have $\mathbf{grad}s(x) = \boldsymbol{w}(x)$. That is

$$\text{Range}(\mathbf{grad}) = \text{Ker}(\mathbf{curl}).$$

Next obviously

$$\text{Range}(\mathbf{curl}) \subset \text{Ker}(\text{div}).$$

Conversely suppose $\boldsymbol{v}(x) \in \text{Ker}(\text{div})$, i.e.,

$$\boldsymbol{v}(x) \in \boldsymbol{V}(T), \quad \text{div}\boldsymbol{v}(x) = 0.$$

Let $\boldsymbol{v}(x) = \boldsymbol{v}_1(x) + \mathbf{curl}(b\boldsymbol{p})$, where $\boldsymbol{v}_1(x) \in P_1^3(T)$, $\boldsymbol{p} \in \tilde{P}^{3*}(T)$.

$$\text{div}\boldsymbol{v}(x) = 0 \iff \text{div}\boldsymbol{v}_1(x) = 0.$$

Suppose

$$\boldsymbol{v}_1(x) = (v_1, v_2, v_3), \quad v_i = \beta_{i0} + \sum_{j=1}^{3} \beta_{ij} x_j, \quad 1 \le i \le 3.$$

Then

$$\text{div}\boldsymbol{v}_1(x) = 0 \iff \sum_{i=1}^{3} \beta_{ii} = 0.$$

Taking $\boldsymbol{w}(x) = \boldsymbol{w}_1(x) + \boldsymbol{s}(x) + b\boldsymbol{p}$, where

$$\boldsymbol{w}_1(x) = (\beta_{20}x_3, \beta_{30}x_1, \beta_{10}x_2), \ \ \boldsymbol{s}(x) = \sum_{i=1}^{8} \alpha_i \boldsymbol{q}_i,$$

$$\alpha_1 = \frac{\beta_{12}}{3}, \quad \alpha_2 = \frac{\beta_{13}}{3}, \quad \alpha_3 = \frac{\beta_{21}}{3}, \quad \alpha_4 = \frac{\beta_{23}}{3}, \quad \alpha_5 = \frac{\beta_{31}}{3}, \quad \alpha_6 = \frac{\beta_{32}}{3},$$

we have

$$\mathbf{curl}(\boldsymbol{w}_1(x) + \boldsymbol{s}(x)) = \boldsymbol{v}_1(x), \quad \mathbf{curl}\boldsymbol{w}(x) = \boldsymbol{v}(x).$$

Hence

$$\text{Range}(\mathbf{curl}) = \text{Ker}(\text{div}).$$

(2) Now we prove (5.5). To prove $\mathbf{grad}\Pi_h^S = \Pi_h^W \mathbf{grad}$, it is enough to prove that $\forall s(x) \in S(T)$, $\mathbf{grad}\Pi_h^S$ satisfies the interpolation conditions of $\Pi_h^W$ for $\mathbf{grad}s(x)$, by (5.9), that is to prove that

$$\int_{l_i} \mathbf{grad}\Pi_h^S s \cdot \boldsymbol{t} p \, dl = \int_{l_i} \mathbf{grad}s \cdot \boldsymbol{t} p \, dl, \qquad \forall p \in P_1(l_i), \ 1 \le i \le 6, \tag{5.17a}$$

$$\int_{f_i} \mathbf{grad}\Pi_h^S s \wedge \boldsymbol{n} \, ds = \int_{f_i} \mathbf{grad}s \wedge \boldsymbol{n} ds, \qquad 1 \le i \le 4, \tag{5.17b}$$

$$\int_{f_i} \mathbf{grad}\Pi_h^S s \cdot \boldsymbol{n} \, ds = \int_{f_i} \mathbf{grad}s \cdot \boldsymbol{n} ds, \qquad 1 \le i \le 4, \tag{5.17c}$$

$$\int_{f_i} \mathbf{curlgrad}\Pi_h^S s \wedge \boldsymbol{n} \, ds = \int_{f_i} \mathbf{curlgrad}s \wedge \boldsymbol{n} ds, \qquad 1 \le i \le 4. \tag{5.17d}$$

In fact, by the interpolation conditions of $\Pi_h^S$ according to (5.5), we have

$$\begin{aligned} \int_{l_i} \mathbf{grad}\Pi_h^S s \cdot \boldsymbol{t} p \, dl &= \int_{l_i} \frac{\partial \Pi_h^S s}{\partial t} p \, dl = \Pi_h^S sp\Big|_{l_i} - \int_{l_i} \Pi_h^S s \frac{\partial p}{\partial t} \, dl \\ &\overset{(5.7)}{=} sp|_{l_i} - \int_{l_i} s \frac{\partial p}{\partial t} \, dl = \int_{l_i} \mathbf{grad}s \cdot \boldsymbol{t} p \, dl, \qquad \forall p \in P_1(l_i), \ 1 \le i \le 4. \end{aligned}$$

That is (5.17a) is proved. For $\boldsymbol{n}=(1,0,0)$,

$$\int_f \mathbf{grad}\Pi_h^S s\wedge \boldsymbol{n}\,ds=\int_f (0,\frac{\partial \Pi_h^S s}{\partial x_3},-\frac{\partial \Pi_h^S}{\partial x_2})\,ds$$
$$=\int_{\partial f}(0,\Pi_h^S s n_3,-\Pi_h^S s n_2)\,dl \overset{(5.7)}{=} \int_{\partial f}(0,sn_3,-sn_2)\,dl=\int_f \mathbf{grad}s\wedge \boldsymbol{n}\,dl.$$

In the same way, for other $\boldsymbol{n}$, (5.17b) is proved. Furthermore, note that

$$\int_{f_i}\mathbf{grad}\Pi_h^S s\cdot \boldsymbol{n}\,ds=\int_{f_i}\frac{\partial \Pi_h^S s}{\partial n}\,ds \overset{(5.7)}{=} \int_{f_i}\frac{\partial s}{\partial n}\,ds=\int_{f_i}\mathbf{grad}s\cdot \boldsymbol{n}\,ds.$$

Then (5.17c) is proved. Since $\mathbf{curlgrad}=0$, (5.17d) automatically holds.

Next to prove $\mathbf{curl}\Pi_h^W=\Pi_h^V\mathbf{curl}$. It is enough to prove that $\forall \boldsymbol{w}(x)\in \boldsymbol{W}(T)$, $\mathbf{curl}\Pi_h^W\boldsymbol{w}(x)$ satisfies the interpolation conditions of $\Pi_h^V$ for $\mathbf{curl}\boldsymbol{w}(x)$ according to (3.34), that is to prove that

$$\int_{f_i}\mathbf{curl}\Pi_h^W\boldsymbol{w}\cdot\boldsymbol{n}p\,ds=\int_{f_i}\mathbf{curl}\boldsymbol{w}\cdot\boldsymbol{n}p\,ds,\quad \forall p\in P_1(f_i),\ 1\le i\le 4, \tag{5.18a}$$
$$\int_{f_i}\mathbf{curl}\Pi_h^W\boldsymbol{w}\wedge\boldsymbol{n}\,ds=\int_{f_i}\mathbf{curl}\boldsymbol{w}\wedge\boldsymbol{n}\,ds,\quad 1\le i\le 4. \tag{5.18b}$$

First we have

$$\mathbf{curl}\boldsymbol{v}\cdot\boldsymbol{n}=\nabla\wedge\boldsymbol{v}\cdot\boldsymbol{n}=\nabla\cdot(\boldsymbol{v}\wedge\boldsymbol{n})=\mathrm{div}(\boldsymbol{v}\wedge\boldsymbol{n}).$$

By the interpolation conditions of $\Pi_h^W$, according to (5.5), we have

$$\int_{f_i}\mathbf{curl}\Pi_h^W\cdot\boldsymbol{n}p\,ds=\int_{f_i}\mathrm{div}(\Pi_h^W\boldsymbol{w}\wedge\boldsymbol{n})p\,ds=\int_{\partial f_i}\Pi_h^W\boldsymbol{w}\wedge\boldsymbol{n}\cdot\boldsymbol{n},$$
$$-\int_{f_i}\Pi_h^W\boldsymbol{w}\wedge\boldsymbol{n}\cdot\nabla p\,ds \overset{(5.10b)}{=} -\int_{f_i}\boldsymbol{w}\wedge\boldsymbol{n}\cdot\nabla p\,ds=\int_{f_i}\mathbf{curl}\boldsymbol{w}\cdot\boldsymbol{n}p\,ds,\quad 1\le i\le 4,$$

i.e. (5.18a) is proved. Note also

$$\int_{f_i}\mathbf{curl}\Pi_h^W\boldsymbol{w}\wedge\boldsymbol{n}\,ds \overset{(5.10d)}{=} \int_{f_i}\mathbf{curl}\boldsymbol{w}\wedge\boldsymbol{n}\,ds,\quad 1\le i\le 4,$$

i.e. (5.18b) holds.

Last we prove $\mathrm{div}\Pi_h^V=\Pi_h^Q\mathrm{div}$. It is sufficient to prove that for any $\forall\boldsymbol{v}\in\boldsymbol{V}_h$, $\mathrm{div}\Pi_h^V v$ satisfies the interpolation conditions of $\Pi_h^Q$ for $\mathrm{div}\boldsymbol{v}$. According to (3.35), that is to prove that

$$\int_T p\,\mathrm{div}\Pi_h^V\boldsymbol{v}\,dx=\int_T p\,\mathrm{div}\boldsymbol{v}\,dx,\quad \forall p\in P_1(T). \tag{5.19}$$

To obtain (5.19), we observe

$$\int_T p\,\mathrm{div}\Pi_h^V\boldsymbol{v}\,dx=\int_{\partial T}p\Pi_h^V\boldsymbol{v}\cdot\boldsymbol{n}\,ds-\int_T\Pi_h^V\boldsymbol{v}\cdot\nabla p\,dx$$
$$\overset{(4.17)}{=}\int_{\partial T}p\boldsymbol{v}\cdot\boldsymbol{n}\,ds-\int_T\boldsymbol{v}\cdot\nabla p\,dx=\int_T p\,\mathrm{div}\boldsymbol{v}\,dx.$$

This completes the proof of the theorem. □

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## References


[1] D.N. Arnold, Differential complexes and numerical stability, in Proceedings of the International Congress of Mathematicians, Beijing: Higher Education Press, **1**, (2002), 137-157.

[2] D.N. Arnold, R.S. Falk and R. Winther, Finite element exterior calculus, homological techniques, and applications, *Acta, Numer.*, **15** (2006), 1-155.

[3] D.N. Arnold and Qin J, Quadratic velocity/linear pressure Stokes elements, *Advances in Computer methods for partial differential equations-VII IMACS*, **3** (1992), 28-34.

[4] S. Badia and R. Codina, Unified stabilized finite element formulations for the Stokes and the Darcy problems, *SIAM J. Numer. Anal.*, **47** (2009), 1971-2000.

[5] S.C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods (3rd ed.), Springer Verlag, New York, 2008.

[6] F. Brezzi, J Douglas and L.D. Marini, Two families of mixed finite elements for second order elliptic problem, *Numer.Math.*, **47** (1985), 217-235.

[7] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.

[8] Brezzi F, Fortin M, Marini L D. Mixed finite element methods with continuous stress. Math. Mod. Meth. Appl. Sci., 1993, 3: 275-287

[9] E. Burman and P. Hansbo, Stabilized Crouzeix-Raviart element for the Darcy-Stokes problem, *Numer. Methods Partial Differential Eq.*, **21** (2005), 986-997.

[10] H.R. Chen, S.C. Chen and Z.H. Qiao, $C^0$-nonconforming tetrahedral and cuboidelements for the three-dimensional fourth order elliptic problem, *Numer. Math.*, **124** (2013), 99-119.

[11] S.C. Chen, L.N. Dong and Z.H. Qiao, Uniformly convergent $H(\mathrm{div})$-conforming rectangular elements for Darcy-Stokes problem, *Science China (Mathematics)*, **12** (2013), 2723-2736.

[12] S.C. Chen, L.N. Dong and J.K. Zhao, Uniformly Convergent Cubic Nonconforming Element For DarcyCStokes Problem, *J. Sci. Comput*, **1** (2017), 231-251.

[13] M. Crouzeix and P.A. Raviart, Conforming and nonconforming finite element methods for solving the stationary stokes equations, *RAIRO Ser.Rouge*, **7** (1973), 33-75.

[14] D.B. Das, Hydrodynamic modeling for groundwater flow through permeable barriers, *Hydrol. Process*, **16** (2002), 3393-3418.

[15] J. Grieble and M. Klitz, Nomohenization and numerical simulation of flow in geometries with textile microstructures, *Multiscale Model. Sim.*, **8** (2010), 1439-1460.

[16] J. Guzman and M. Neilan M, A family of nonconforming elements for the Brinkman problem, *IMA J. Numer. Anal.*, **32** (2012), 1484-1508.

[17] T. Kara and J. Goldak, Three-dimensional numerical analysis of heat and mass transfer in heat pipes, *Heat Mass Trasfer.*, **43** (2007), 775-785.

[18] J. Konno, Finite Element Methods for Flow in Porous Media, Phd thesis, Aalto University, 2011.

[19] J. Konno and R. Stenberg, Numerical computations with H(div)-finite elements for the Brinkman problem, *Computational Geoscienes*, **16** (2012), 139-158.

[20] J. Konno and R. Stenberg, Analysis of H(div)-conforming finite elements for the Brinkman problem, *Math. Models Methods Appl. Sci.*, **21** (2011), 2227-2248.

[21] C.Y. Liua, W.M. Yinga and J.O. Tan, Flow in the adiabatic section of a heat pipe, *Int. Commun. Heat Mass transfer.*, **16** (1989), 79-88.

[22] K.A. Mardal and X.C. Tai, Winther R. A robust finite element method for Darcy-Stokes flow, *SIAM J. Numer. Anal*, **40** (2002), 1605-1631.

[23] W. Miladi and M. Racila, Mathematical model of fluid flow in an osteon influence of cardiac system, *Computer Methods in Biomechanics and Biomedical Engineering*, **12** (2009), 187-188.

[24] V. Nassehi, Modelling of combined Navier-Stokes and Darcy flows in crossflow membrane filtration, *Chem. Eng. Sci.*, **53** (1998), 1253-1265.

[25] J.C. Nédelec, Mixed finite elements in $R^3$, *Numer. Math.*, **35** (1980), 315-341.

[26] J.C. Nédelec, A new family of mixed finite element in $R^3$, *Numer. Math.*, **50** (1986), 57-81.

[27] J. Qin, On the convergence of some low order mixed finite elements for incompressible fluids, PhD. Thesis, Penn State University, 1994.

[28] P.A. Raviart and J. Thomas, A mixed finite element method for 2-nd order elliptic problems. In I. Galligani & E. Magenes, editors, Mathematical Aspects of the Finite Elements Method, Lectures Notes in Math., Berlin, spriger, **606** (1977), 292-315.

[29] L.R. Scott, Vogelius M. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials, *Math. Model. Num. Anal.*, **19** (1985), 111-143.

[30] X.C. Tai, Winther R. A discrete de Rham complex with enhanced smoothness, *CALCOLO*, **43** (2006), 287-306.

[31] X.P. Xie, J.C. Xu and G.R. Xue, Uniformly-stable finite element methods for Darcy-Stokes-Brinkman models, *J. Comput. Math.*, **26** (2008), 437-455.

[32] X.J. Xu and S.Y. Zhang, A new divergence-free interpolation operator with applications to the Darcy-Stokes-Brinkman equations, *SIAM J. Sci. Comput.*, **32** (2010), 855-874.

[33] G.R. Xue, Numerical Methods for Multiphysics, Multiphase, and Multicomponent Models for Fuel Cells, PhD. Thesis, Pennsylvania State University, 2008.

[34] S.Y. Zhang, A new family of stable mixed finite elements for the 3D Stokes equations, *Math. Comput.*, **74** (2005), 543-554.

[35] S. Zhang S, X.P. Xie and Y. Chen, Low order nonconforming rectangular finite element methods for Darcy-Stokes problems, *J. Comput. Math.*, **27** (2009), 400-424.