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ON THE GENERALIZED DETERIORATED POSITIVE SEMI-DEFINITE AND SKEW-HERMITIAN SPLITTING PRECONDITIONER*

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Abstract

For nonsymmetric saddle point problems, Huang et al. in [Numer. Algor. 75 (2017), pp. 1161-1191] established a generalized variant of the deteriorated positive semi-definite and skew-Hermitian splitting (GVDPSS) preconditioner to expedite the convergence speed of the Krylov subspace iteration methods like the GMRES method. In this paper, some new convergence properties as well as some new numerical results are presented to validate the theoretical results.

Mathematics subject classification: 65F10, 65N22.
Key words: Saddle point problem, Preconditioner, Nonsymmetric, Symmetric, Positive definite, Krylov subspace method.

1. Introduction

Consider the solution of large sparse saddle point problems of the form

$$\mathcal{A}u \equiv \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, the matrix $B \in \mathbb{R}^{m\times n}$ is of full row rank with $m \leq n$, B^T denotes the transpose of the matrix B. Moreover, $x, f \in \mathbb{R}^n$ and $y, g \in \mathbb{R}^m$. We are especially interested in cases that the matrix A is symmetric positive definite or nonsymmetric with positive definite symmetric part (i.e., A is real positive). When $A = A^T$, the linear system (1.1) is called the symmetric saddle point problem and, when $A \neq A^T$, it is called the nonsymmetric saddle point problem. According to Lemma 1.1 in [7] the matrix $\mathcal{A}$ is nonsingular.

In the last decade, there has been tremendous efforts to develop fast solution methods for solving the saddle point problems. As is well-known, Krylov subspace methods [15] are the most effective methods for solving the saddle point problems of the form (1.1). But the convergence rate of these methods depend closely on the eigenvalues and the eigenvectors of the

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coefficient matrix [1,15] and they tend to converge slowly when are applied to the saddle point problem (1.1). In general, favourable rates of convergence of Krylov subspace methods are often incorporated with a well-clustered spectrum of the preconditioned matrices (away from zero). Therefore, many kinds of preconditioners have been studied in the literature for saddle point matrix, e.g., HSS-based preconditioners [2, 4, 5, 7], block diagonal preconditioners [17], block triangular preconditioners [3,17], shift-splitting preconditioners [6,10], and so on.

Zhang and Gu in [18] established a variant of the deteriorated positive semi-definite and skew-Hermitian splitting (VDPSS) preconditioner as follows

$$\mathcal{M}_{VDPSS} = \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \alpha I \end{pmatrix}, \tag{1.2}$$

for the problem (1.1). Recently, Huang et al. in [13] proposed a generalization of the VDPSS (GVDPSS) preconditioner of the form

$$\mathcal{P}_{GVDPSS} = \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix}. \tag{1.3}$$

The difference between $\mathcal{P}_{GVDPSS}$ and $\mathcal{A}$ is given by

$$\mathcal{R}_{GVDPSS} = \mathcal{P}_{GVDPSS} - \mathcal{A} = \begin{pmatrix} 0 & \frac{1}{\alpha}AB^T - B^T \\ 0 & \beta I \end{pmatrix}.$$

It follows from the latter equation that as $\beta \longrightarrow 0^+$, the (2, 2)-block of $\mathcal{R}_{GVDPSS}$ tends to zero matrix and as $\alpha \longrightarrow +\infty$, the (1, 2)-block of $\mathcal{R}$ tends to $-B^T$. So, it seems that the GVDPSS preconditioner with proper parameters α and β is more closer to the coefficient matrix $\mathcal{A}$ than the VDPSS preconditioner due to the independence of the parameters and, as a result, the corresponding preconditioned matrix will have a well-clustered spectrum.

It can be seen that by choosing different values for the parameters α and β, the GVDPSS preconditioner coincides with some existing preconditioners such as the RHSS preconditioner [11], the REHSS preconditioner [16], the RDPSS preconditioner [9] and the VDPSS preconditioner [18].

The GVDPSS preconditioner can be derived from the GVDPSS iteration method. Huang et al. have presented the convergence properties of the GVDPSS iteration method and the spectral properties of the corresponding preconditioned matrix in [13], but nothing about the optimal values of the involved parameters. In this paper, we present new convergence properties and the optimal parameters, which minimize the spectral radius of the iteration matrix of the GVDPSS iteration method.

2. New Convergence Results for the GVDPSS Iteration Method

The GVDPSS preconditioner $\mathcal{P}_{GVDPSS}$ can be induced by a fixed-point iteration, which is based on the following splitting of the coefficient matrix $\mathcal{A}$:

$$\mathcal{A} = \mathcal{P}_{GVDPSS} - \mathcal{R}_{GVDPSS} \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{\alpha}AB^T - B^T \\ 0 & \beta I \end{pmatrix}. \tag{2.1}$$

Based on this splitting, the GVDPSS iteration method can be constructed as

$$\begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix} u^{(k+1)} = \begin{pmatrix} 0 & \frac{1}{\alpha}AB^T - B^T \\ 0 & \beta I \end{pmatrix} u^{(k)} + \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{2.2}$$

where $\alpha > 0$ and $\beta > 0$ are the iteration parameters and $u^{(0)}$ is an initial guess. Note that the iterative method can also be written as a fixed point form $u^{k+1} = \Gamma u^k + c$, where

$$\Gamma = \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix}^{-1} \begin{pmatrix} 0 & \frac{1}{\alpha}AB^T - B^T \\ 0 & \beta I \end{pmatrix}, \tag{2.3}$$

is the iteration matrix and $c = \mathcal{P}^{-1}b$.

Theorem 2.1. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite and Γ be the iteration matrix* (2.3) *of the iteration method* (2.2)*. Then, $\rho(\Gamma) < 1$ if one of the following two conditions holds true.*

(i) *If $\alpha > 0$ and $\beta \geq \max\{\lambda_{\max}(Q), 0\}$, where $\lambda_{\max}(Q)$ is the largest eigenvalue of the matrix $Q \equiv B(\frac{1}{2}A^{-1} - \frac{1}{\alpha}I)B^T$.*

(ii) *If $\beta \geq 0$ and $\alpha < 2\lambda_{\min}(A)$, where $\lambda_{\min}(A)$ is the smallest eigenvalue of A.*

Proof. From Lemma 2.1 in [13], we get

$$\begin{aligned}
\Gamma &= \mathcal{P}^{-1}\mathcal{R} = \mathcal{P}^{-1}(\mathcal{P} - \mathcal{A}) = I - \mathcal{P}^{-1}\mathcal{A} \\
&= I - \begin{pmatrix} A^{-1} - \frac{1}{\alpha}B^TS^{-1}BA^{-1} & -\frac{1}{\alpha}B^TS^{-1} \\ S^{-1}BA^{-1} & S^{-1} \end{pmatrix} \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \\
&= I - \begin{pmatrix} I & A^{-1}B^T - \frac{1}{\alpha}B^TS^{-1}BA^{-1}B^T \\ 0 & S^{-1}BA^{-1}B^T \end{pmatrix} \\
&= \begin{pmatrix} 0 & \tilde{A} \\ 0 & I - \hat{A} \end{pmatrix},
\end{aligned} \tag{2.4}$$

where $S = \beta I + \frac{1}{\alpha}BB^T$, $\tilde{A} = -A^{-1}B^T + \frac{1}{\alpha}B^TS^{-1}BA^{-1}B^T$ and $\hat{A} = S^{-1}BA^{-1}B^T$. Therefore, if λ is an eigenvalue of the iteration matrix Γ, then $\lambda = 0$ or $\lambda = 1 - \mu$, where μ is the solution of the eigenvalue problem

$$S^{-1}BA^{-1}B^Tx = \mu x. \tag{2.5}$$

Since $S = \beta I + \frac{1}{\alpha}BB^T$, Eq. (2.5) is equivalent to

$$BA^{-1}B^Tx = \mu\left(\beta I + \frac{1}{\alpha}BB^T\right)x. \tag{2.6}$$

It is clear that $x \neq 0$. Without loss of generality, we suppose that $\|x\|_2 = 1$. Since $B^Tx \neq 0$, multiplying the above equation by x^* on both sides yields

$$\mu = \frac{x^*BA^{-1}B^Tx}{\beta + \frac{1}{\alpha}x^*BB^Tx} > 0.$$

Therefore, $|\lambda| < 1$ if and only if

$$\frac{x^*BA^{-1}B^Tx}{\beta + \frac{1}{\alpha}x^*BB^Tx} < 2,$$

which is equivalent to

$$\beta > \frac{1}{2}x^*BA^{-1}B^Tx - \frac{1}{\alpha}x^*BB^Tx. \tag{2.7}$$

To prove part (i), let α be an arbitrary positive constant. From above discussion, it is easy to know that a sufficient condition to have $|\lambda| < 1$ is that

$$\beta > \lambda_{\max}(Q) = \max_{\|y\|_2=1} y^*Qy > x^*Qx = \frac{1}{2}x^*BA^{-1}B^Tx - \frac{1}{\alpha}x^*BB^Tx,$$

which completes the proof of (i).

Now, we prove the part (ii). If the right-hand side of the inequality (2.7) is less than zero then for every $\beta \geq 0$ the inequality (2.7) holds true and, as a result, we get $\rho(\Gamma) < 1$. Hence, the parameter α must be chosen such a way that

$$\frac{1}{2}x^* BA^{-1}B^T x - \frac{1}{\alpha}x^* BB^T x < 0,$$

which is equivalent to

$$\alpha < \frac{2y^*y}{y^*A^{-1}y},$$

where $y = B^T x$. For the above relation to be held, it is enough to have

$$\alpha < 2\lambda_{\min}(A) = \frac{2}{\lambda_{\max}(A^{-1})} = 2\min_{y\neq 0}\frac{y^*y}{y^*A^{-1}y} < \frac{2y^*y}{y^*A^{-1}y}, \tag{2.8}$$

which completes the proof. □

Now, we consider the GVDPSS iteration method (2.2) from another point of view. Let $\omega \geq 0$ be a fixed constant. Consider the parameters $\alpha > 0$ and $\beta \geq 0$ such that $\omega = \alpha\beta$. In this case, the iteration matrix Γ of the GVDPSS iteration method (2.2) is given by

$$\Gamma = \begin{pmatrix} 0 & A^{-1}B^T - \frac{1}{\alpha}B^T S^{-1}BA^{-1}B^T \\ 0 & I - \alpha(\omega I + BB^T)^{-1}BA^{-1}B^T \end{pmatrix}. \tag{2.9}$$

It follows from Eq. (2.9) that the spectral radius of the iteration matrix Γ is

$$\rho(\Gamma) = \max_{1\leq i\leq m}|1 - \alpha\mu_i|, \tag{2.10}$$

where μ_i, $i = 1, \ldots, m$, are the eigenvalues of the matrix $(\omega I + BB^T)^{-1}BA^{-1}B^T$. Since $(\omega I + BB^T)^{-1}BA^{-1}B^T$ is similar to $(\omega I + BB^T)^{-\frac{1}{2}}BA^{-1}B^T(\omega I + BB^T)^{-\frac{1}{2}}$, then, in fact, the spectral radius (2.10) is the same as that of the stationary Richardson iteration when applied to the following linear system

$$(\omega I + BB^T)^{-\frac{1}{2}}BA^{-1}B^T(\omega I + BB^T)^{-\frac{1}{2}}x = b.$$

Therefore, it is expected that the convergence properties of the Richardson iteration method and the GVDPSS iteration method (2.2) with $\beta = \omega/\alpha$ are the same. Then, using the above results and an argument like Theorem 3.1 in [11], we can deduce the following theorem.

Theorem 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite and Γ be the iteration matrix* (2.3) *of the GVDPSS iteration method* (2.2). *Let $\omega \geq 0$ be a constant and consider the parameters $\alpha > 0$ and $\beta \geq 0$ such that $\omega = \alpha\beta$. Then,*

$$\rho(\Gamma) = \max_{1\leq i\leq m}|1 - \alpha\mu_i|,$$

where μ_i, $i = 1, \ldots, m$, are the eigenvalues of $(\omega I + BB^T)^{-1}BA^{-1}B^T$. Let μ_m and μ_1 be the smallest and largest eigenvalues of $(\omega I + BB^T)^{-1}BA^{-1}B^T$, respectively. For every $\omega \geq 0$, if

$$0 < \alpha < \frac{1}{\mu_1},$$

and $\beta = \omega/\alpha$ then the GVDPSS iteration method is convergent. The optimal value of α which minimizes the spectral radius $\rho(\Gamma)$ is given by $\alpha_{\rm opt} = 2/(\mu_1 + \mu_m)$, which yields $\beta_{\rm opt} = \omega/\alpha_{\rm opt}$. The corresponding optimal convergence factor is given by

$$\rho(\Gamma) = \frac{\mu_1 - \mu_m}{\mu_1 + \mu_m}.$$

Remark 2.1. If $\omega = 0$ then $\beta = 0$ and, as a result, the preconditioner $\mathcal{P}_{GVDPSS}$ reduces to the RHSS preconditioner. Hence, the corresponding optimal value α^* is exactly the same as that of the RHSS preconditioner (Theorem 3.1 in [11]).

Theorem 2.3. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite, the preconditioner $\mathcal{P}_{GVDPSS}$ be defined in* (1.3) *and $\alpha > 0$ and $\beta \geq 0$. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalues of $\mathcal{A}$, respectively. Let σ_m and σ_1 be the smallest and largest singular values of matrix B, respectively. Then, the preconditioned matrix $\mathcal{P}^{-1}_{GVDPSS}\mathcal{A}$ has an eigenvalue 1 of algebraic multiplicity at least n. The remaining eigenvalues are positive real and the solution of the generalized eigenvalue problem:*

$$BA^{-1}B^Tx = \mu(\beta I + \frac{1}{\alpha}BB^T)x.$$

In addition, the non-unit eigenvalues μ satisfy

$$\frac{\alpha}{\lambda_{\max}(A)(1+\alpha\beta)} \leq \mu \leq \frac{\alpha}{\lambda_{\min}(A)(1+\alpha\beta)}. \tag{2.11}$$

Proof. It follows from Eq. (2.4) that

$$\mathcal{P}^{-1}\mathcal{A} = \begin{pmatrix} I & A^{-1}B^T - \frac{1}{\alpha}B^TS^{-1}BA^{-1}B^T \\ 0 & S^{-1}BA^{-1}B^T \end{pmatrix} = \begin{pmatrix} I & \tilde{A} \\ 0 & \hat{A} \end{pmatrix}, \tag{2.12}$$

where $S = \beta I + \frac{1}{\alpha}BB^T$. It follows from Eq. (2.12) that the preconditioned matrix $\mathcal{P}^{-1}_{GVDPSS}\mathcal{A}$ has an eigenvalue 1 of algebraic multiplicity at least n, and the remaining eigenvalues are the solution of the following generalized eigenvalue problem

$$BA^{-1}B^Tx = \mu\left(\beta I + \frac{1}{\alpha}BB^T\right)x. \tag{2.13}$$

Since BB^T and $BA^{-1}B^T$ are symmetric and positive definite, $\beta \geq 0$ and $\alpha > 0$, the solution of the generalized eigenvalue problem (2.13) is positive real. This completes the proof of the first part of the theorem. Let (μ, x) be an eigenpair of the generalized eigenvalue problem (2.13). If we set $\hat{x} = x/\|B^Tx\|_2$ then $(\mu, \hat{x})$ is also an eigenpair of (2.13). As seen in the proof of Theorem 2.1, we can write

$$\mu = \frac{\hat{x}^*BA^{-1}B^T\hat{x}}{\beta + \frac{1}{\alpha}\hat{x}^*BB^T\hat{x}} = \frac{\hat{x}^*BA^{-1}B^T\hat{x}}{\beta + \frac{1}{\alpha}}. \tag{2.14}$$

According to Theorem 1.22 of [15], we have

$$\hat{x}^*BA^{-1}B^T\hat{x} \leq \lambda_{\max}(A^{-1})\hat{x}^*BB^T\hat{x} = \frac{1}{\lambda_{\min}(A)},$$
$$\frac{1}{\lambda_{\max}(A)} = \lambda_{\min}(A^{-1})\hat{x}^*BB^T\hat{x} \leq \hat{x}^*BA^{-1}B^T\hat{x},$$

that prove (2.11). □

Remark 2.2. It immediately follows from Eq. (2.11) that if $\beta \longrightarrow 0^+$ then the non-unit eigenvalues μ of the preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ satisfy

$$\frac{\alpha}{\lambda_{\max}(A)} \leq \mu \leq \frac{\alpha}{\lambda_{\min}(A)},$$

and if $\alpha \longrightarrow +\infty$, then

$$\frac{1}{\beta\lambda_{\max}(A)} \leq \mu \leq \frac{1}{\beta\lambda_{\min}(A)}.$$

Theorem 2.4. *Let $A \in \mathbb{R}^{n\times n}$ be nonsymmetric and positive definite and Γ be the iteration matrix (2.3) of the GVDPSS iteration method (2.2). Let $\omega \geq 0$ be a fixed constant and consider the parameters $\alpha > 0$ and $\beta \geq 0$ such that $\omega = \alpha\beta$. Then,*

$$\rho(\Gamma) = \max_{1\leq j\leq m} |1 - \alpha(\gamma_j + i\eta_j)|,$$

where $\gamma_j + i\eta_j$, $j = 1, \ldots, m$, are the eigenvalues of $(\omega I + BB^T)^{-1}BA^{-1}B^T$ and $i = \sqrt{-1}$. Let γ_1 and γ_m, and η_1 and η_m be the upper and lower bounds of the real, the absolute values of the imaginary parts of the eigenvalues of $(\omega I + BB^T)^{-1}BA^{-1}B^T$, respectively. For every $\omega \geq 0$, if

$$0 < \alpha < \begin{cases} \frac{2\gamma_m}{\gamma_m^2+\eta_1^2}, & \text{if } \eta_1 \geq \sqrt{\gamma_1\gamma_m}, \\ \\ \frac{2\gamma_1}{\gamma_1^2+\eta_1^2}, & \text{if } \eta_1 < \sqrt{\gamma_1\gamma_m}, \end{cases}$$

and $\beta = \omega/\alpha$ then the GVDPSS iteration method is convergent. The optimal value of α which minimizes the spectral radius $\rho(\Gamma)$ is given by

$$\alpha_{\text{opt}} = \begin{cases} \frac{\gamma_m}{\gamma_m^2+\eta_1^2}, & \text{if } \eta_1 \geq \sqrt{\gamma_1\gamma_m}, \\ \\ \frac{2}{\gamma_1+\gamma_m}, & \text{if } \eta_1 < \sqrt{\gamma_1\gamma_m}, \end{cases}$$

which yields $\beta_{\text{opt}} = \omega/\alpha_{\text{opt}}$. The corresponding optimal convergence factor is given by

$$\rho_{\text{opt}}(\gamma) = \begin{cases} \frac{\eta_1}{\sqrt{\gamma_m^2+\eta_1^2}}, & \text{if } \eta_1 \geq \sqrt{\gamma_1\gamma_m}, \\ \\ \frac{\sqrt{(\gamma_1-\gamma_m)^2+4\eta_1^2}}{\gamma_1+\gamma_m}, & \text{if } \eta_1 < \sqrt{\gamma_1\gamma_m}. \end{cases}$$

Proof. As seen before, when $\beta = \omega/\alpha$, the iteration matrix Γ is of the form (2.9). It shows that the spectral radius of the iteration matrix Γ is

$$\rho(\Gamma) = \max_{1\leq j\leq m} |1 - \alpha(\gamma_j + i\eta_j)|, \tag{2.15}$$

where $\gamma_j + i\eta_j$, $j = 1, \ldots, m$, are the eigenvalues of $(\omega I + BB^T)^{-1}BA^{-1}B^T$. The rest of the proof is similar to that of Theorem 2.2 in [9] and is omitted. □

Remark 2.3. If $\omega = 0$, then $\beta = 0$ and, as a result, the preconditioner $\mathcal{P}_{GVDPSS}$ reduces to the RDPSS preconditioner. Hence, the corresponding optimal value α^* is exactly the same as that of the RDPSS preconditioner (Theorem 2.2 in [9]).

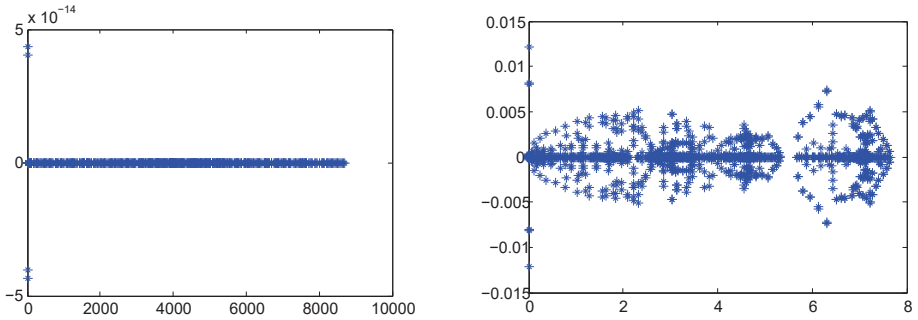

Fig. 3.1. Eigenvalue distribution of the saddle point matrix $\mathcal{A}$ for Example 3.1 with $q = 32$ (left) and for Example 3.2 with 32×32 grid (right).

3. Numerical Experiments

In this section, we use two examples to validate the theoretical analysis of the previous section and illustrate the feasibility and effectiveness of the GVDPSS preconditioner (1.3) when it is applied to accelerate the convergence rate of Krylov subspace iteration methods such as GMRES for solving saddle point problems (1.1), from the point of view of both the number of iterations (denoted by IT) and and the elapsed CPU time in seconds (denoted by CPU). As mentioned before, the GVDPSS preconditioner (1.3) covers the RHSS preconditioner [11] and the REHSS preconditioner [16] when it is applied to solve the symmetric saddle point problems as well as the DPSS preconditioner [14] and the VDPSS preconditioner [18] when it is applied to solve the nonsymmetric saddle point problems. The first and second examples lead to a symmetric and nonsymmetric saddle point problem, respectively. The Krylov subspace methods such as GMRES incorporated with RHSS preconditioner, REHSS preconditioner, the GVDPSS preconditioner, the PHSS preconditioner [5] and the AHSS preconditioner [2] are applied to solve the symmetric saddle point problem in the first example, and then their results are compared with each other. In addition, the RPDSS and VDPSS preconditioners as well as the GVDPSS preconditioner are applied to solve the nonsymmetric saddle point problem in the second example, and then their results are compared with each other.

All tests are performed in MATLAB on a Laptop with Intel Core i7 CPU 1.8 GHz, 6GB RAM. In all the tests, the right-hand side vector b is chosen so that the exact solution of the saddle point problem (1.1) is a vector of all ones. Besides, all runs are started from a null vector, and are terminated if the current iteration satisfies $\|r_k\|_2 \leq 10^{-6}\|r_0\|_2$, where $r_k = b - \mathcal{A}u^{(k)}$ is the residual at the kth iteration. At each step of applying the preconditioners, it is required to solve the sub-linear systems which can be done by direct methods. In MATLAB, this corresponds to computing the Cholesky or LU factorization in combination with AMD or column AMD reordering.

Example 3.1. ([5]) We consider the Stokes problem

$$-\mu\triangle\mathbf{u} + \nabla p = \tilde{f}, \qquad \text{in } \Omega, \tag{3.1a}$$

$$\nabla.\mathbf{u} = \tilde{g}, \qquad \text{in } \Omega, \tag{3.1b}$$

$$\mathbf{u} = 0, \qquad \text{on } \partial\Omega, \tag{3.1c}$$

$$\int_\Omega p dx = 0, \tag{3.1d}$$

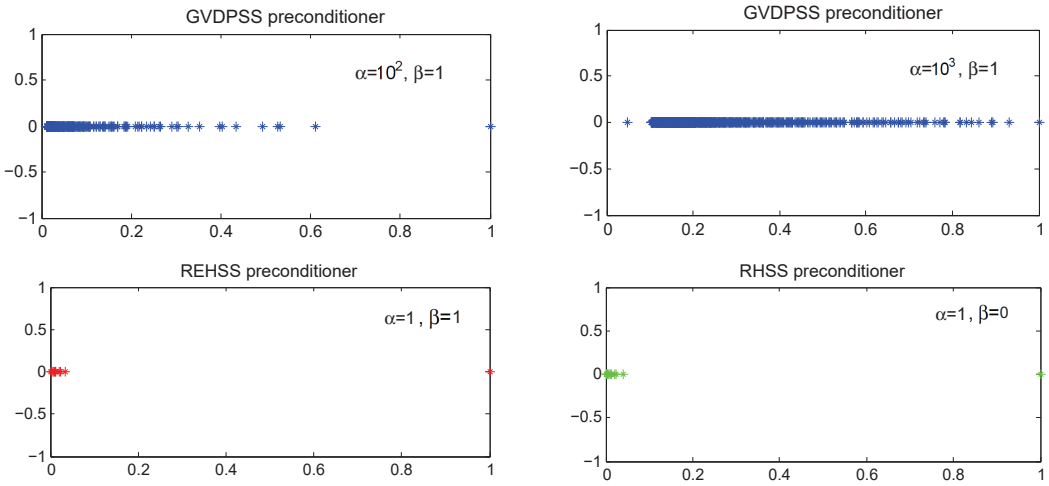

Fig. 3.2. Eigenvalue distribution of the preconditioned matrix for Example 3.1 ($q = 32$).

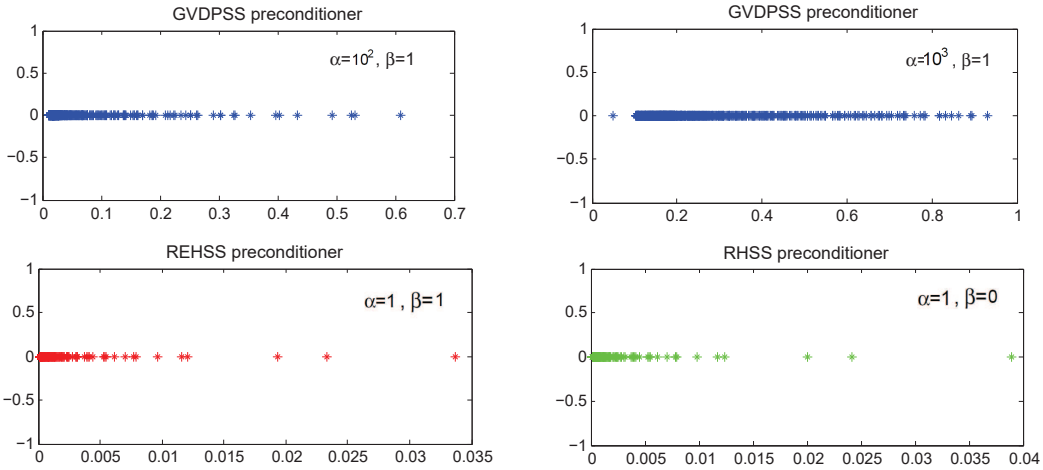

Fig. 3.3. Non-unit eigenvalue distribution of the preconditioned matrix for Example 3.1 ($q = 32$).

where $\Omega = (-1, 1) \times (-1, 1)$, $\partial\Omega$ is the boundary of the domain Ω, $\triangle$ is the componentwise Laplace operator, $\mathbf{u}$ is a vector-valued function representing the velocity, and p is a scalar function representing the pressure. By discretizing (3.1) by the upwind scheme, we obtain the system of linear equations (1.1) where

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2q^2 \times 2q^2}, \qquad B^T = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2q^2 \times q^2},$$

$$T = \frac{\mu}{h^2}.\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{q \times q}, \qquad F = \frac{1}{h}.\text{tridiag}(-1, 1, 0) \in \mathbb{R}^{q \times q},$$

with $\otimes$ being the Kronecker product symbol and $h = 1/(q+1)$ the discretization meshsize. For this example, we have $n = 2q^2$ and $m = q^2$. Hence, the total number of variables is $m+n = 3q^2$. Here, we consider $\mu = 1$.

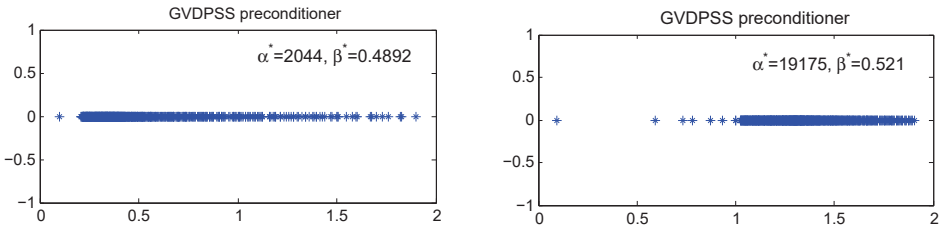

Fig. 3.4. Eigenvalue distribution of the preconditioned matrix with optimal parameters for Example 3.1 ($q = 32$).

From [2], we know that the AHSS preconditioner is given by

$$\mathcal{M}(\alpha,\beta) = \begin{pmatrix} \frac{\alpha+1}{2}A & \frac{\alpha+1}{2\alpha}B^T \\ -\frac{1}{2}B & \frac{\beta}{2}C \end{pmatrix},$$

where C is an arbitrary Hermitian positive definite matrix. It is noted that the PHSS preconditioner is a special case of the AHSS preconditioner when $\alpha = \beta$. A natural choice of the matrix C is $C = B\widehat{A}^{-1}B^T$, where $\widehat{A}$ is a good approximation of the matrix block A. For this example, we take $\widehat{A} = \frac{2\mu}{h^2}I + I \otimes T$ the block-diagonal matrix A. Based on Theorem 3.2 in [5] and [2], we can obtain the optimal parameters α^* and (α^*, β^*) of the PHSS and AHSS iteration methods, respectively. In Table 3.1, these optimal parameters and the corresponding numerical results with respect to iteration step and CPU time, for various problem sizes q, are listed. In Table 3.2, we list the optimal parameters α^* and β^* of the GVDPSS preconditioner, determined by the formula given in Theorem 2.2, and the corresponding numerical results with respect to iteration step and CPU time for different values of ω and q. For each q, it is seen that by changing ω, the changes of the corresponding optimal parameter β^* in comparison with optimal parameter α^* is significantly slow (for new preconditioner). It should be noted that if $\omega = 0$ then $\beta = 0$ and, as a result, the preconditioner $\mathcal{P}_{GVDPSS}$ reduces to the RHSS preconditioner which its results were shown in boldface type. It is also noticed that α^* corresponding to $\omega = 0$ plays the role of the optimal parameter of the RHSS preconditioner. As seen, the preconditioner $\mathcal{P}_{GVDPSS}$ with optimal parameters is more effective than the RHSS, PHSS and AHSS preconditioners in terms of both the iteration steps and the CPU time. Moreover, by increasing the value of ω, the preconditioner $\mathcal{P}_{GVDPSS}$ with corresponding optimal parameters has a considerable reduction in the number of the iteration steps.

To further investigate the influence of the new parameter in the convergence speed of the

Table 3.1: The optimal parameters of the PHSS and AHSS preconditioners and the corresponding numerical results for Example 3.1 with different q.

Method	q	16	32	48	64
PHSS-GMRES	α^*	1.8718	2.5657	3.1113	3.5753
	IT	46	94	149	209
	CPU	0.106	5.049	51.01	266
AHSS-GMRES	α^*	1.5026	1.9482	2.3115	2.6256
	β^*	2.3317	3.3789	4.1879	4.8685
	IT	43	84	131	184
	CPU	0.097	4.424	43.57	235

Table 3.2: The optimal parameters of the GVDPSS preconditioner and the numerical results for the corresponding preconditioned GMRES method for Example 3.1 with different ω and q.

q	ω	α^*	β^*	IT	CPU
16	**0**	**49.25**	**0**	**23**	**0.0178**
	1	56.91	0.0176	23	0.01795
	10^1	104.32	0.0959	21	0.01602
	10^2	307.61	0.3251	15	0.01195
	10^3	1966	0.5086	10	0.0094
	10^4	18473	0.5413	9	0.0087
32	**0**	**51.19**	**0**	**36**	**0.1151**
	1	59.18	0.0169	36	0.1097
	10^1	107.34	0.0932	34	0.10290
	10^2	321.8	0.3108	26	0.0769
	10^3	2044	0.4892	15	0.0470
	10^4	19175	0.521	10	0.0343
48	**0**	**51.82**	**0**	**47**	**0.5210**
	1	59.90	0.0167	46	0.5153
	10^1	108.06	0.0925	44	0.4892
	10^2	324.5	0.3081	36	0.3931
	10^3	2076	0.4817	19	0.2146
	10^4	19461	0.5138	11	0.1338
64	**0**	**52.13**	**0**	**56**	**1.950**
	1	60.25	0.0166	56	1.926
	10^1	108.36	0.0923	54	1.884
	10^2	325.48	0.3072	45	1.561
	10^3	2093	0.4776	23	0.799
	10^4	19616	0.5098	11	0.406

preconditioned GMRES method, we present the numerical results for different values of α and β in Table 3.3 with respect to iteration step and CPU time (in parentheses). It should be noted that when $\beta = 0$ the preconditioner $\mathcal{P}_{GVDPSS}$ reduces to the RHSS preconditioner which its results were shown in boldface type, and when $\alpha = 1$ it coincides with the REHSS preconditioner which its results were underlined. Table 3.3 indicates that the GVDPSS preconditioner is more effective than the RHSS, REHSS, PHSS and AHSS preconditioners. Specially, for $\beta \geq 1$, the use of the GVDPSS preconditioner with some large values of α results in a considerable reduction in the number of iteration steps.

We plot the eigenvalue distribution of the matrix $\mathcal{A}$ in Fig 3.1 and that of the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ in Fig 3.2. As mentioned, if $\beta = 0$ then the $\mathcal{P}_{GVDPSS}$ preconditioner reduces to the RHSS preconditioner and if $\alpha = 1$ then the $\mathcal{P}_{GVDPSS}$ preconditioner reduces to the REHSS preconditioner. The non-unit eigenvalue distribution of the preconditioned matrices in Fig 3.2 are drawn in Fig 3.3. Fig 3.4 depicts the eigenvalue distribution of the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ with optimal parameters. These figures show that the use of the preconditioner $\mathcal{P}_{GVDPSS}$ leads to a well-clustered spectrum away from zero which in turn results in a faster convergence of GMRES. In addition, Fig 3.2 verifies our statement in Theorem 2.3 that the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ has at least n eigenvalues 1 and the remaining non-unit eigenvalues are positive real.

Example 3.2. Consider the Oseen problem which is obtained from the linearization of the following steady-state Navier-Stokes equation by the Picard iteration with suitable boundary

Table 3.3: Numerical results for the GMRES method incorporated with the GVDPSS preconditioner for Example 3.1 with different q.

q	α \ β	0	10^{-1}	1	10	10^2
16	10^{-1}	**27(0.0189)**	27(0.0199)	27(0.0184)	27(0.0193)	26(0.0199)
	1	**25(0.0178)**	25(0.0193)	25(0.0182)	24(0.0191)	19(0.0145)
	10	**24(0.0170)**	24(0.0166)	23(0.0163)	18(0.0142)	11(0.0099)
	10^2	**22(0.0160)**	21(0.0150)	16(0.0125)	11(0.0096)	9(0.0085)
	10^3	**21(0.0166)**	15(0.0122)	9(0.0086)	7(0.0077)	8(0.0081)
	α \ β	0	10^{-1}	1	10	10^2
32	10^{-1}	**43(0.1393)**	43(0.1393)	43(0.1359)	43(0.1336)	41(0.1263)
	1	**41(0.1243)**	41(0.1243)	41(0.1242)	39(0.1230)	32(0.0994)
	10	**38(0.1142)**	38(0.1142)	37(0.1176)	30(0.1014)	18(0.0578)
	10^2	**34(0.1059)**	34(0.1059)	28(0.0866)	16(0.0515)	10(0.0369)
	10^3	**26(0.0811)**	26(0.0811)	15(0.0557)	10(0.0362)	8(0.0291)
	α \ β	0	10^{-1}	1	10	10^2
48	10^{-1}	**56(0.6560)**	56(0.6456)	56(0.7553)	56(0.7061)	54(0.6389)
	1	**53(0.6176)**	53(0.6288)	53(0.6974)	51(0.6233)	44(0.5325)
	10	**50(0.5898)**	50(0.6024)	48(0.5910)	42(0.5024)	23(0.3170)
	10^2	**46(0.5609)**	45(0.5408)	38(0.4549)	22(0.2665)	11(0.1349)
	10^3	**44(0.5493)**	36(0.4043)	19(0.2428)	11(0.1366)	9(0.1194)
	α \ β	0	10^{-1}	1	10	10^2
64	10^{-1}	**68(2.550)**	68(2.495)	68(2.488)	68(2.501)	66(2.449)
	1	**64(2.334)**	64(2.347)	64(2.340)	62(2.277)	54(1.962)
	10	**60(2.154)**	60(2.162)	58(2.119)	51(1.842)	28(1.005)
	10^2	**55(1.962)**	54(1.952)	48(1.743)	26(0.919)	13(0.484)
	10^3	**53(1.921)**	44(1.590)	23(0.840)	12(0.444)	9(0.347)

condition on $\partial\Omega$

$$\begin{cases} -\nu\triangle\mathbf{u} + (\mathbf{w}.\nabla)\mathbf{u} + \nabla p = f, & \text{in } \Omega, \\ \text{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = g, & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $\nu > 0$, $\triangle$, ∇, div, $\mathbf{u}$, and p stand for the Laplace operator, the gradient operator, the divergence, velocity and pressure of the fluid, respectively. Here the vector field $\mathbf{w}$ is the approximation of $\mathbf{u}$ from the previous Picard iteration.

Many approximation schemes can be applied to discretize the Oseen problem (3.2) leading to a saddle point system of type (1.1). We consider a leaky two-dimensional lid-driven cavity problem discretized by Q2-P1 finite element on uniform grids on the unit square. The test problem was generated by using the IFISS software package written by Elman et al. [12]. We use the viscosity value $\nu = 1$ to generate linear systems corresponding to 16×16, 32×32, 64×64 and 128×128 meshes. The numerical results for optimal parameters α^* and β^*, determined by the formula given in Theorem 2.4, corresponding to different valuses of ω on 16×16, 32×32, 64×64 uniform grids are given in Table 3.4. For each grid, we can see that by changing ω, the corresponding optimal values β^* is almost constant while the optimal values α^* changes increasingly. In addition, it is observed that the values of optimal parameters of the grid 32×32

Table 3.4: The optimal parameters of the GVDPSS preconditioner and the numerical results for the corresponding preconditioned GMRES method for Example 3.2 with different uniform grids.

Grids	ω	α^*	β^*	IT	CPU
	0	**0.1909**	**0**	**27**	**0.0557**
	0.1	3.846	0.0260	20	0.0396
16×16	1	33.018	0.0303	20	0.0397
	5	161.28	0.0310	20	0.0399
	10	321.13	0.0311	20	0.0401
	0	**0.0488**	**0**	**42**	**0.2778**
	0.1	13.156	0.008	21	0.1631
32×32	1	128.54	0.008	20	0.1452
	5	640	0.008	20	0.1491
	10	1280	0.008	20	0.1517
	0	**0.0123**	**0**	**66**	**2.893**
	0.1	14.339	0.007	20	0.925
64×64	1	138.04	0.007	20	0.933
	5	689	0.007	20	0.933
	10	1379	0.007	20	0.906

are significantly close to that of the grid size 64×64 . It should be noted that if $\omega = 0$ then $\beta = 0$ and, as a result, the preconditioner $\mathcal{P}_{GVDPSS}$ reduces to the RDPSS preconditioner which its results were shown in boldface type. It is also noticed that α^* corresponding to $\omega = 0$ plays the role of the optimal parameter of the RDPSS preconditioner. As seen, the performance of the preconditioner $\mathcal{P}_{GVDPSS}$ with optimal parameters is better than that of the RDPSS preconditioner in terms of both iteration steps and CPU time. Moreover, it is known that the efficiency of the preconditioner $\mathcal{P}_{GVDPSS}$ with optimal parameters corresponding to nonzero ω is often the same.

Further, in Table 3.5, we present the numerical results with respect to iteration step and CPU time (in parentheses) for different values of α and β to analyze the influence of the GVDPSS parameter in the convergence speed of the preconditioned GMRES method. It should be noted that when $\beta = 0$ the preconditioner $\mathcal{P}_{GVDPSS}$ reduces to the RDPSS preconditioner which its results were shown in boldface type, and when $\alpha = \beta$ it reduces to the VRDPSS preconditioner which its results were underlined. This table indicates that the performance of the new preconditioner $\mathcal{P}_{GVDPSS}$ is often better than that of the RDPSS and VRDPSS preconditioners. Specially in the case of small values of β, the use of preconditioner $\mathcal{P}_{GVDPSS}$ with some large value of α results in a considerable reduction in the number of iteration steps, while the other two preconditioners are difficult to implement efficiently. For large value of β, the GVDPSS preconditioner has the best performance and is no longer sensitive to the value of the parameter α, in the sense that the number of iteration does not change drastically. It is also noted that the GVDPSS preconditioner is not competitive in the case of large β.

We plot the eigenvalue distribution of the matrix $\mathcal{A}$ in Fig 3.1, the eigenvalue distribution of the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ in Fig 3.5, and the eigenvalue distribution of the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ with optimal parameters in Fig 3.6. As mentioned, if $\beta = 0$ then the preconditioner $\mathcal{P}_{GVDPSS}$ reduces to the RDPSS preconditioner and if $\alpha = \beta$ then the preconditioner $\mathcal{P}_{GVDPSS}$ reduces to the VDPSS preconditioner. It is evident that the preconditioner matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ is of a well-clustered spectrum around $(1, 0)$ away from zero. In addition, these figures verify the our statement in Theorem 2.3 that the preconditioned

Table 3.5: Numerical results for the GMRES method incorporated with the GVDPSS preconditioner for Example 3.2 with different uniform grids.

Grids	α \ β	0	10^{-3}	10^{-2}	10^{-1}	1	10^2
16×16	10^{-3}	**31(0.0579)**	31(0.0541)	31(0.0511)	31(0.0531)	30(0.0568)	26(0.0437)
	10^{-2}	**29(0.0473)**	29(0.0522)	29(0.0545)	28(0.0463)	28(0.0478)	22(0.0406)
	10^{-1}	**27(0.0454)**	27(0.0498)	26(0.0446)	27(0.0501)	23(0.0423)	21(0.0388)
	1	**26(0.0447)**	24(0.0428)	25(0.0432)	21(0.0389)	20(0.0374)	21(0.0404)
	10^2	**31(0.0498)**	25(0.0454)	23(0.0405)	18(0.0357)	19(0.0394)	21(0.0389)
	α \ β	0	10^{-3}	10^{-2}	10^{-1}	1	10^2
32×32	10^{-3}	**48(0.3312)**	48(0.3218)	47(0.3258)	47(0.3405)	50(0.3308)	26(0.1854)
	10^{-2}	**44(0.3037)**	44(0.2948)	44(0.3038)	47(0.3254)	34(0.2439)	22(0.1586)
	10^{-1}	**41(0.2866)**	41(0.2827)	43(0.2879)	31(0.2077)	23(0.1682)	22(0.1586)
	1	**40(0.2905)**	42(0.2807)	28(0.1982)	22(0.1579)	19(0.1393)	22(0.1597)
	10^2	**48(0.3346)**	24(0.1729)	21(0.1455)	18(0.1329)	19(0.1348)	22(0.1558)
	α \ β	0	10^{-3}	10^{-2}	10^{-1}	1	10^2
64×64	10^{-3}	**71(3.336)**	71(3.311)	76(3.489)	78(3.598)	52(2.386)	24(1.166)
	10^{-2}	**67(3.063)**	71(3.304)	72(3.299)	49(2.247)	29(1.368)	20(0.984)
	10^{-1}	**62(2.854)**	66(3.031)	45(2.047)	27(1.312)	21(1.127)	20(0.9856)
	1	**63(2.893)**	42(1.951)	25(1.178)	20(0.979)	18(0.906)	20(0.984)
	10^2	**76(3.692)**	23(1.099)	20(0.997)	17(0.854)	18(0.898)	20(0.978)
	α \ β	0	10^{-3}	10^{-2}	10^{-1}	1	10^2
128×128	10^{-3}	**108(28.91)**	120(31.55)	121(31.63)	78(20.45)	20(5.64)	8(2.82)
	10^{-2}	**100(26.08)**	113(29.71)	74(19.65)	19(5.53)	12(3.73)	7(2.62)
	10^{-1}	**96(25.04)**	66(17.10)	17(4.93)	11(3.60)	7(2.57)	7(2.61)
	1	**97(25.31)**	16(4.70)	10(3.31)	7(2.62)	6(2.35)	7(2.59)
	10^2	**118(30.97)**	10(3.22)	7(2.62)	6(2.35)	6(2.38)	7(2.72)

matrix $\mathcal{P}^{-1}_{GVDPSS}\mathcal{A}$ has at least n eigenvalues 1.

4. Conclusion

We have presented new convergence properties of the GVDPSS method and the optimal parameters, which minimize the spectral radius of the iteration matrix of the GVDPSS iteration method. The GVDPSS preconditioner, which involves two parameters, covers the relaxed versions of the HSS preconditioner [11, 16] when it is applied to solve the symmetric saddle point problems as well as the RDPSS preconditioner [9] and the VDPSS preconditioner [18] when it is applied to solve the nonsymmetric saddle point problems. Some numerical examples have been presented to validate the theoretical analysis and show the effectiveness of the method. Numerical performances have shown that the GVDPSS preconditioner is superior to the RHSS, REHSS and RDPSS preconditioners at accelerating the convergence rates of Krylov subspace iteration methods such as GMRES. The performance of the GVDPSS preconditioner is better than that of the VDPSS preconditioner for small value of the parameter β, but it is

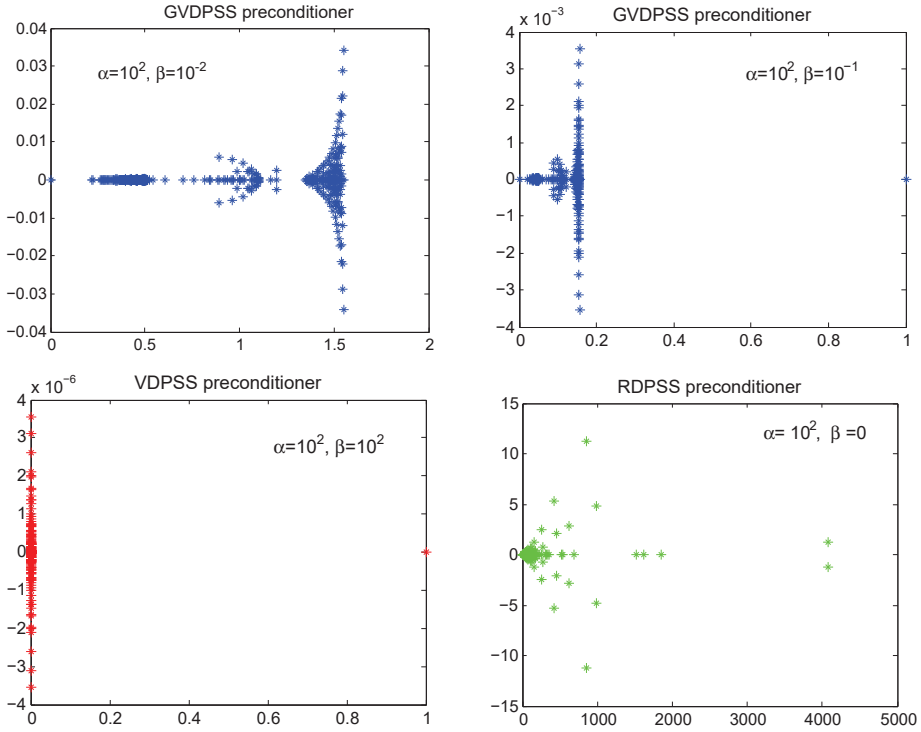

Fig. 3.5. Eigenvalue distribution of the preconditioned matrix for Example 3.2 (32×32 grid).

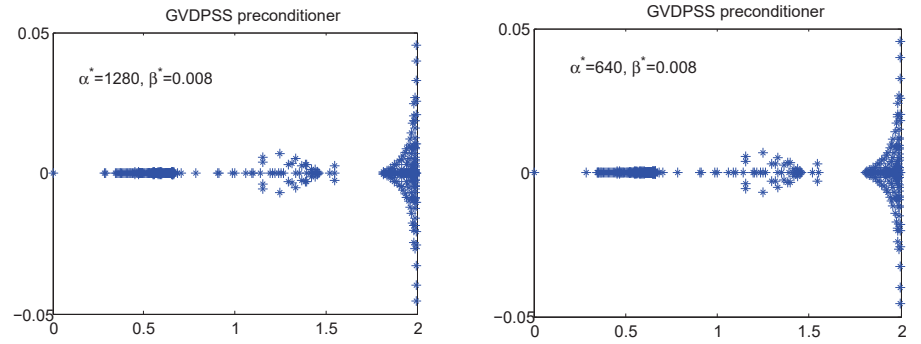

Fig. 3.6. Eigenvalue distribution of the preconditioned matrix with optimal parameters for Example 3.2 (32×32 grid).

not competitive in the case of large β.

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