Journal of Computational Mathematics
Vol.37, No.1, 2019, 18–32.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1707-m2016-0730

ON THE GENERALIZED DETERIORATED POSITIVE SEMI-DEFINITE AND SKEW-HERMITIAN SPLITTING PRECONDITIONER*

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Abstract

For nonsymmetric saddle point problems, Huang et al. in [Numer. Algor. 75 (2017), pp. 1161-1191] established a generalized variant of the deteriorated positive semi-definite and skew-Hermitian splitting (GVDPSS) preconditioner to expedite the convergence speed of the Krylov subspace iteration methods like the GMRES method. In this paper, some new convergence properties as well as some new numerical results are presented to validate the theoretical results.

Mathematics subject classification: 65F10, 65N22.
Key words: Saddle point problem, Preconditioner, Nonsymmetric, Symmetric, Positive definite, Krylov subspace method.

1. Introduction

Consider the solution of large sparse saddle point problems of the form

$$\mathcal{A}u \equiv \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, the matrix $B \in \mathbb{R}^{m\times n}$ is of full row rank with $m \le n$, B^T denotes the transpose of the matrix B. Moreover, $x, f \in \mathbb{R}^n$ and $y, g \in \mathbb{R}^m$. We are especially interested in cases that the matrix A is symmetric positive definite or nonsymmetric with positive definite symmetric part (i.e., A is real positive). When $A = A^T$, the linear system (1.1) is called the symmetric saddle point problem and, when $A \neq A^T$, it is called the nonsymmetric saddle point problem. According to Lemma 1.1 in [7] the matrix $\mathcal{A}$ is nonsingular.

In the last decade, there has been tremendous efforts to develop fast solution methods for solving the saddle point problems. As is well-known, Krylov subspace methods [15] are the most effective methods for solving the saddle point problems of the form (1.1). But the convergence rate of these methods depend closely on the eigenvalues and the eigenvectors of the

* Received August 28, 2016 / Revised version received May 31, 2017 / Accepted July 10, 2017 /
Published online August 21, 2018 /

coefficient matrix [1,15] and they tend to converge slowly when are applied to the saddle point problem (1.1). In general, favourable rates of convergence of Krylov subspace methods are often incorporated with a well-clustered spectrum of the preconditioned matrices (away from zero). Therefore, many kinds of preconditioners have been studied in the literature for saddle point matrix, e.g., HSS-based preconditioners [2,4,5,7], block diagonal preconditioners [17], block triangular preconditioners [3,17], shift-splitting preconditioners [6,10], and so on.

Zhang and Gu in [18] established a variant of the deteriorated positive semi-definite and skew-Hermitian splitting (VDPSS) preconditioner as follows

$$\mathcal{M}_{VDPSS} = \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \alpha I \end{pmatrix}, \tag{1.2}$$

for the problem (1.1). Recently, Huang et al. in [13] proposed a generalization of the VDPSS (GVDPSS) preconditioner of the form

$$\mathcal{P}_{GVDPSS} = \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix}. \tag{1.3}$$

The difference between $\mathcal{P}_{GVDPSS}$ and $\mathcal{A}$ is given by

$$\mathcal{R}_{GVDPSS} = \mathcal{P}_{GVDPSS} - \mathcal{A} = \begin{pmatrix} 0 & \frac{1}{\alpha}AB^T - B^T \\ 0 & \beta I \end{pmatrix}.$$

It follows from the latter equation that as $\beta \longrightarrow 0^+$, the (2, 2)-block of $\mathcal{R}_{GVDPSS}$ tends to zero matrix and as $\alpha \longrightarrow +\infty$, the (1, 2)-block of $\mathcal{R}$ tends to $-B^T$. So, it seems that the GVDPSS preconditioner with proper parameters α and β is more closer to the coefficient matrix $\mathcal{A}$ than the VDPSS preconditioner due to the independence of the parameters and, as a result, the corresponding preconditioned matrix will have a well-clustered spectrum.

It can be seen that by choosing different values for the parameters α and β, the GVDPSS preconditioner coincides with some existing preconditioners such as the RHSS preconditioner [11], the REHSS preconditioner [16], the RDPSS preconditioner [9] and the VDPSS preconditioner [18].

The GVDPSS preconditioner can be derived from the GVDPSS iteration method. Huang et al. have presented the convergence properties of the GVDPSS iteration method and the spectral properties of the corresponding preconditioned matrix in [13], but nothing about the optimal values of the involved parameters. In this paper, we present new convergence properties and the optimal parameters, which minimize the spectral radius of the iteration matrix of the GVDPSS iteration method.

2. New Convergence Results for the GVDPSS Iteration Method

The GVDPSS preconditioner $\mathcal{P}_{GVDPSS}$ can be induced by a fixed-point iteration, which is based on the following splitting of the coefficient matrix $\mathcal{A}$:

$$\mathcal{A} = \mathcal{P}_{GVDPSS} - \mathcal{R}_{GVDPSS} \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{\alpha}AB^T - B^T \\ 0 & \beta I \end{pmatrix}. \tag{2.1}$$

Based on this splitting, the GVDPSS iteration method can be constructed as

$$\begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix} u^{(k+1)} = \begin{pmatrix} 0 & \frac{1}{\alpha}AB^T - B^T \\ 0 & \beta I \end{pmatrix} u^{(k)} + \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{2.2}$$